Lab 11 – Propositional Logic

1. Glossary

Interpretation = evaluation of a formula for its sentences.
Truth table = table containing all interpretations of a formula.
A formula with n propositions will have 2^n interpretations.
Realizable formula = a formula which can be true.
Tautology = a formula which is always true (true for all its interpretations.
Contradiction = formula which is never true.
Contingency = formula which is neither a tautology, nor a contradiction.
Equivalent formulas = formulas which have the same truth table.

2. Logic Operators

We define a sentence as a statement that can be either true (T) or false (F), but not both at the same time. In propositional logic, we can build more complex sentences based on simple propositions using logical operators, just as in ordinary language we can build more complex structures using key words such as and, or, if, etc.

Definition: Logic negation (NOT)

We consider proposition a. Its negation, not a, denoted by ¬a, is defined by the table of truth below:

a	¬a
T	F
F	T

Remark: ¬(¬a) = a

Definition: Logic disjunction (OR)

We consider the propositions a and b. Their disjunction, a or b, denoted by a ∨ b, is defined by the truth table below:

a	b	a ∨ b
T	T	T
T	F	T
F	T	T
F	F	F

Definition: Logic conjunction (AND)

We consider the propositions a and b. Their conjunction, a and b, denoted by $a \wedge b$, is defined by the truth table below:

a	b	$a \wedge b$
T	T	T
T	F	F
F	T	F
F	F	F

Remark: Distributivity

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Remark: Identity and absorption

$a \vee F = a$ (identity)

$a \wedge T = a$ (identity)

$a \vee (a \wedge b) = a$ (absorption)

$a \wedge (a \vee b) = a$ (absorption)

Definition: Implication

We consider the propositions a and b. Implication describes a relationship of the type “if a then b”, denoted by $a \rightarrow b$, is defined by the truth table below:

a	b	$a \rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

Remark: $a \rightarrow b = \neg a \vee b$

3. De Morgan's laws

We consider the propositions a and b:

$\neg(a \vee b) = \neg a \wedge \neg b$

$\neg(a \wedge b) = \neg a \vee \neg b$

Demonstration (Optional)

To demonstrate de Morgan's rules and show that the two formulas are equivalent, we will build the truth tables:

a	b	$a \vee b$	$\neg(a \vee b)$	$\neg a$	$\neg b$	$\neg a \wedge \neg b$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

a	b	$a \wedge b$	$\neg(a \wedge b)$	$\neg a$	$\neg b$	$\neg a \vee \neg b$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

4. Conjunctive Normal Form (CNF)

Conjunctive Normal Form (CNF) = A conjunction ($\wedge$) of clauses
Clause = Disjunction ($\vee$) of literals (propositions of negations of propositions)

Example:

$(a \vee b \vee c) \wedge (\neg a \vee \neg b \vee \neg c) \wedge a$ is in CNF.

$(a \vee b \vee c)$ is in CNF.

$(a \wedge b \wedge c) \vee (\neg a \wedge \neg b \wedge \neg c)$ is NOT CNF.

$(a \vee b \wedge c)$ is NOT CNF.

In order to bring a formula to the CNF we have to carry the negation inside to the proposition (using de Morgan's rules) and bring the disjunction ∨ inside the conjunction ∧ using the property of distributivity.

Attention: Before doing this, do not forget to rewrite the implications using the formula
$a \rightarrow b = \neg a \lor b$
Observation: In certain cases, the absorption and identity properties can also be used for to simplify the formula.

Warmup exercise

Bring the following formula to the CNF, create the truth table and specify formula type:
$f = a \lor (b \land c)$

Solution:
$f = a \lor (b \land c)$
$f = (a \lor b) \land (a \lor c)$ – using distributivity

a	b	c	a ∨ b	a ∨ c	f = (a ∨ b) ∧ (a ∨ c)
F	F	F	F	F	F
F	F	T	F	T	F
F	T	F	T	F	F
F	T	T	T	T	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

The formula is realizable.
You can check the result on Wolfram Alpha, at this link:
https://www.wolframalpha.com/input?i=CNF+%28a+or+%28b+and+c%29%29

Example exercise

Bring the following formula to the CNF, create the truth table and specify formula type:

$f = (p \land q) \lor (p \land \neg q)$

Solution:

$f = (p \land q) \lor (p \land \neg q)$

$f = ((p \land q) \lor p) \land ((p \land q) \lor \neg q)$

$f = ((p \lor p) \land (p \lor q)) \land ((p \lor \neg q) \land (\neg q \lor q))$ -eliminate redundant parentheses

$f = (p \lor p) \land (p \lor q) \land (p \lor \neg q) \land (\neg q \lor q)$ – we use $p \lor p = p$ and $\neg q \lor q = T$

$f = p \land (p \lor q) \land (p \lor \neg q) \land T$ -we know that $a \land T = a$

$f = p \land (p \lor q) \land (p \lor \neg q)$

$f = p$ – absorption

p	q	$p \lor q$	$\neg q$	$p \lor \neg q$	$f = p \land (p \lor q) \land (p \lor \neg q)$
F	F	F	T	T	F
F	T	T	F	F	F
T	F	T	T	T	T
T	T	T	F	T	T

The formula is realizable.

Observation: The first and last columns are identical.

You can check the result on Wolfram Alpha, at this link:

https://www.wolframalpha.com/input?i=CNF+%28%28p+and+q%29+or+%28p+and+not+q%29%29

Example exercise

Bring the following formula to the CNF, create the truth table and specify formula type:

$f = (a \lor b) \to \neg(a \lor c)$

Solution:

$f = (a \lor b) \to \neg(a \lor c)$

$f = \neg(a \lor b) \lor \neg(a \lor c)$

$f = (\neg a \land \neg b) \lor (\neg a \land \neg c)$

$f = (\neg a \lor (\neg a \land \neg c)) \land (\neg b \lor (\neg a \land \neg c))$

$f = \neg a \land (\neg b \lor (\neg a \land \neg c))$

$f = \neg a \land (\neg b \lor \neg a) \land (\neg b \lor \neg c)$

$f = \neg a \land (\neg b \lor \neg c)$

a	b	c	¬a	¬b	¬c	¬b ∨ ¬c	f = ¬a ∧ (¬b ∨ ¬c)
F	F	F	T	T	T	T	T
F	F	T	T	T	F	T	T
F	T	F	T	F	T	T	T
F	T	T	T	F	F	F	F
T	F	F	F	T	T	T	F
T	F	T	F	T	F	T	F
T	T	F	F	F	T	T	F
T	T	T	F	F	F	F	F

The formula is realizable.
You can check the result on Wolfram Alpha, at this link:
https://www.wolframalpha.com/input?i=CNF+%28%28a+or+b%29+implies+not+%28a+or+c%29%29

Example exercise

Bring the following formula to the CNF, create the truth table and specify formula type:

$f = \neg(p \to q) \vee ((r \vee s) \to (q \vee t)) \vee (\neg p \to \neg v)$

Solution:

$f = \neg(p \to q) \vee ((r \vee s) \to (q \vee t)) \vee (\neg p \to \neg v)$

$f = \neg(\neg p \vee q) \vee (\neg(r \vee s) \vee (q \vee t)) \vee (p \vee \neg v)$

$f = (p \wedge \neg q) \vee (\neg r \wedge \neg s) \vee q \vee t \vee p \vee \neg v$

$f = (p \wedge \neg q) \vee ((\neg r \vee q \vee t \vee p \vee \neg v) \wedge (\neg s \vee q \vee t \vee p \vee \neg v))$

$f = (p \vee \neg r \vee q \vee t \vee \neg v) \wedge (\neg s \vee q \vee t \vee p \vee \neg v) \wedge$
$(p \vee \neg r \vee q \vee t \vee \neg v \vee \neg q) \wedge (\neg s \vee q \vee t \vee p \vee \neg v \vee \neg q)$

$f = (p \vee \neg r \vee q \vee t \vee \neg v) \wedge (\neg s \vee q \vee t \vee p \vee \neg v) \wedge T \wedge T$

$f = (p \vee \neg r \vee q \vee t \vee \neg v) \wedge (\neg s \vee q \vee t \vee p \vee \neg v)$

We used $q \vee \neg q = T$

The formula is realizable for $p = T$, $r = F$, $q = T$, $t = T$, $v = F$, $s = F$.

You can check the result on Wolfram Alpha, at this link:
https://www.wolframalpha.com/input?i=CNF+%28not%28p+implies+q%29or+%28%28r+or+s%29implies%28q+or+t%29%29or%28not+p+implies+not+v%29%29

Example exercise

Bring the following formula to the CNF, create the truth table and specify formula type:

$f = \neg((a \wedge b) \vee ((a \rightarrow (b \wedge c)) \rightarrow c))$

Solution:

$f = \neg((a \wedge b) \vee ((a \rightarrow (b \wedge c)) \rightarrow c))$

$f = \neg((a \wedge b) \vee ((\neg a \vee (b \wedge c)) \rightarrow c))$

$f = \neg((a \wedge b) \vee (\neg(\neg a \vee (b \wedge c)) \vee c))$

$f = (\neg(a \wedge b) \wedge \neg(\neg(\neg a \vee (b \wedge c)) \vee c))$

$f = ((\neg a \vee \neg b) \wedge (\neg\neg(\neg a \vee (b \wedge c)) \wedge \neg c))$

$f = (\neg a \vee \neg b) \wedge (\neg a \vee (b \wedge c)) \wedge \neg c$ -we eliminate redundant parentheses

$f = (\neg a \vee \neg b) \wedge ((\neg a \vee b) \wedge (\neg a \vee c)) \wedge \neg c$

$f = (\neg a \vee \neg b) \wedge (\neg a \vee b) \wedge (\neg a \vee c) \wedge \neg c$ (CNF, can be simplified further)

$f = (((\neg a \vee \neg b) \wedge \neg a) \vee ((\neg a \vee \neg b) \wedge b)) \wedge (\neg a \vee c) \wedge \neg c$

$f = (\neg a \vee ((\neg a \vee \neg b) \wedge b))) \wedge (\neg a \vee c) \wedge \neg c$ -absorption

$f = (\neg a \vee ((\neg a \vee \neg b) \wedge b))) \wedge \neg c$ - absorption

$f = (\neg a \vee ((\neg a \wedge b) \vee (\neg b \wedge b))) \wedge \neg c$

$f = (\neg a \vee ((\neg a \wedge b) \vee F)) \wedge \neg c$

$f = (\neg a \vee ((\neg a \wedge b))) \wedge \neg c$

$f = \neg a \wedge \neg c$ - absorption

a	c	¬a	¬c	f
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	F	F	F

The formula is realizable.

You can check the result on Wolfram Alpha, at this link:
https://www.wolframalpha.com/input?i=CNF+%28not%28%28a+and+b%29or+%28not%28not+a+or+%28%28b+and+c%29%29%29or+c%29%29%29

5. Binary Decision Diagram (BDD)

To build a BDD, choose a proposition p from a formula f and calculate $f_{p=T}$ s i $f_{p=F}$ (replace p in the formula first with T then with F). If as a result of the evaluation you have obtained T, F or an elementary proposition (for example q or ¬q) then you have finished evaluating the formula. If you have obtained a compound formula, recursively apply the algorithm (choose a new proposition, etc.). When you have finished evaluating all the formulas, draw all the propositions as nodes in a diagram. In the diagram there will be two nodes 0 and 1. 0 corresponds to False, 1 corresponds to True. For each evaluated proposition, add the decision to the diagram and the result of its evaluation.

Example exercise

Create the BDD for the following formula:

$f = (a \land b) \lor (c \land d)$

Solution:

We choose proposition a.

$f_{a=T} = (T \land b) \lor (c \land d) = b \lor (c \land d) = f^1$ (we note the newly obtained formula with f^1. 1 does not represent power, only an index)

$f_{a=F} = (F \land b) \lor (c \land d) = c \land d = f^2$

Because we have not obtained base cases, we well evaluate f^1 and f^2.

For f^1 we choose proposition b.

$f^1{}_{b=T} = T \lor (c \land d) = T$ (base case)

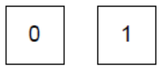
$f^1{}_{b=F} = F \lor (c \land d) = c \land d = f^2$

We still need to evaluate f^2. We choose proposition c.

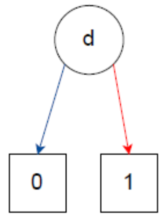
$f^2{}_{c=T} = T \land d = d$ (base case)

$f^2{}_{c=F} = F \land d = F$ (base case)

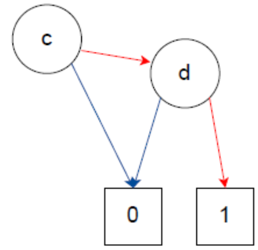
We have evaluated all cases, so now we can build the BDD. For this example, the process will be presented step by step.



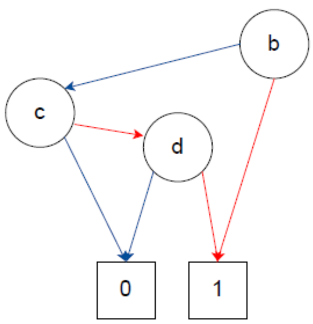
Initially we have nodes 0 and 1



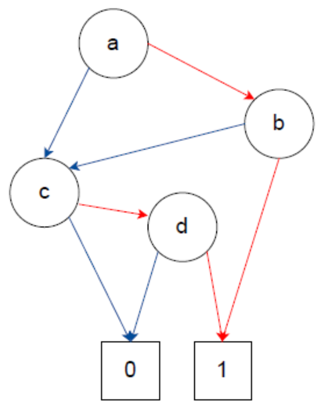
We add d (the result of f2 when c=T). If d is True then the result is True, otherwise the result if False. (red is True, blue is False)



We add c. For c=T the expression depends on d. For c=F, the expression is False.



We add b.



Finally, we add a.

Example exercise

Create the BDD for the following formula:

$f = (a \lor b) \to \neg(a \lor c)$

Solution:

$f = (a \lor b) \to \neg(a \lor c)$

$f = \neg(a \lor b) \lor \neg(a \lor c)$

We choose proposition a.

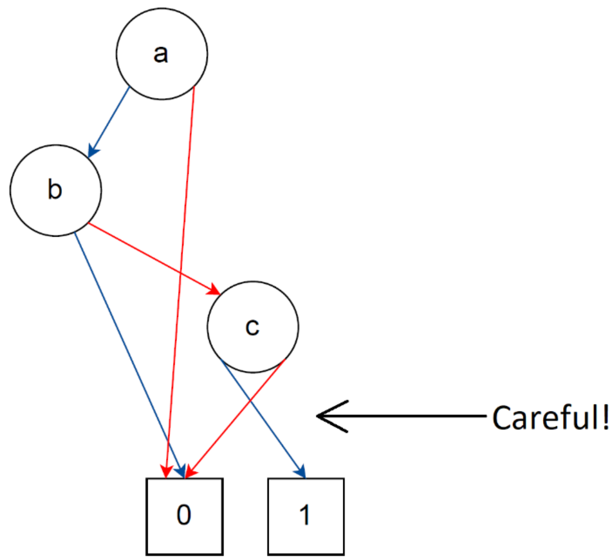
$f_{a=T} = \neg(T \lor b) \lor \neg(T \lor c) = F$ (base case)

$f_{a=F} = \neg(F \lor b) \lor \neg(F \lor c) = \neg b \lor \neg c = f^1$

We will evaluate f_1. For f_1 we choose proposition b.

$f^1_{b=T} = \neg T \lor \neg c = \neg c$ (base case. Attention! The arrows need to be reversed because we have a negation.)

$f^1_{b=F} = \neg F \lor \neg c = T$



6. Propositional logic in Python (Optional)

Can be found on Campus Virtual.

7. Homework

For this homework, upload a single file (.doc, .docx, .pdf, etc.). If you solve it on paper, scan the pages and upload the PDF file. No not upload images. Solve at least 2 of the 4 exerciese.

Bring to conjunctive normal form (CNF), create the truth table, specify the formula type and draw the binary decision diagram (BDD) for the following expressions:

1. $\neg((c \rightarrow \neg b) \land (a \rightarrow \neg c))$
2. $\neg((b \lor c) \rightarrow \neg(\neg a \lor \neg b) \lor \neg b)$
3. $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c))$
4. $\neg((a \land b) \land (c \lor (\neg(a \rightarrow (b \land c)))))$