Fuzzy sets. Operations with fuzzy sets

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Outline

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Crisp (classic) sets, fuzzy sets

- Given an *universe of discourse* (*crisp*) X
- For a classic (crisp) set $A \subset X$, for each element $x \in X$, either $x \in A$ or $x \notin A$.
- For the set A it can be defined a characteristic function $\nu_A : X \to \{0, 1\}$, with $\nu_A(x) = 1$ iff (if and only if) $x \in A$ and $\nu_A(x) = 0$ iff $x \notin A$
- For a fuzzy set $\tilde{A}$, an element $x \in X$ belongs to the fuzzy set $\tilde{A} \subset X$ **in a certain degree**
- The characteristic function of a crisp set will be extended to the *membership function* of a fuzzy set, which can take values in the real numbers interval $[0, 1]$

Definitions

Definition

"If X is a collection of objects" (named the *universe of discourse*) "denoted generically by x, then a fuzzy set $\tilde{A} \subset X$ is a set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

where $\mu_{\tilde{A}}(x) : X \to [0, 1]$ is called *membership function* or *degree of membership* (also, degree of compatibility or degree of truth). of x in A" (Zimmermann [Zim91])

If the interval of real numbers [0, 1] is replaced with the discrete set $\{0, 1\}$, then the fuzzy set $\tilde{A}$ becomes a classic (crisp) set.

Fuzzy sets. Examples of fuzzy sets

- Fuzzy sets can be discrete or continuous
- The interval $[0, 1]$ can be extended to $[0, k]$, where $k > 0$
- It is possible to define fuzzy sets on more complex structures than intervals or real numbers, e.g. $\mathbb{L}$-*fuzzy sets*, where $\mathbb{L}$ is a partially ordered set (see chapter 3, Extensions of fuzzy sets)
- Example of discrete fuzzy set (Zimmermann [Zim91]):
  - MF: comfortable house for a 4 person family as a function of the number of bedrooms:
  - The universe discourse: $X = \{1, 2, \ldots, 10\}$
  - $\tilde{A} \subset X$ will be $\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$

Examples of fuzzy sets (cnt'd)

Example of continuous fuzzy set: real numbers close to 10

- $X = \mathbb{R}$ (the set of real numbers)
- The membership function of the fuzzy set $\tilde{A} \subset \mathbb{R}$:

$$\mu_{\tilde{A}}(x) = \frac{1}{1 + (x - 10)^2} \tag{1}$$

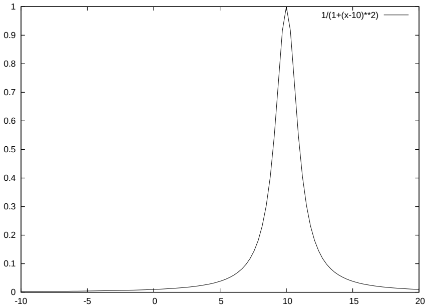

Figure 1: $\tilde{A}$ with $\mu_{\tilde{A}}(x) = \frac{1}{1+(x-10)^2}$

Examples of fuzzy sets (cnt'd)

Example of a continuous fuzzy set: real numbers considerably larger than 11

- $X = \mathbb{R}$ (the set of real numbers)
- The membership function of the fuzzy set: $\tilde{B} \subset \mathbb{R}$:

$$\mu_{\tilde{B}}(x) = \begin{cases} \frac{(x-11)^2}{1+(x-11)^2} & \text{if } x \geq 11 \\ 0, & \text{if } x < 11 \end{cases} \tag{2}$$

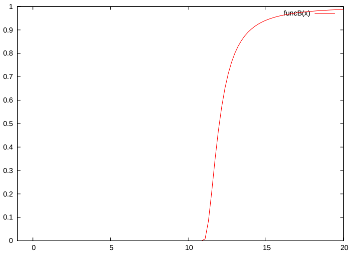

Figure 2: $\tilde{B}$ with $\mu_{\tilde{B}}(x)$

Notations for fuzzy sets

1. Pairs *(element, value)* for discrete fuzzy sets (like in the example with the comfortable house), respectively *(generic element, membership function)* for continuous fuzzy sets: e.g. $(x, \mu_{\tilde{A}}(x))$
2. Solely by stating the membership function (for continuous fuzzy sets)
3. As a "sum" for discrete fuzzy sets, respectively "integral" for continuous fuzzy sets (this notation may create confusions !!):

$$\tilde{A} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i} = \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \ldots + \frac{\mu_{\tilde{A}}(x_n)}{x_n}$$

$$\tilde{A} = \int \frac{\mu_{\tilde{A}}(x)}{x}$$

Caution, there are neither sums nor integrals here, these are only notations !!!

Outline

Properties (characteristics) of fuzzy sets: normal fuzzy sets

1. Normal fuzzy sets
   - A fuzzy set is called *normal* if $\sup_x \mu_{\tilde{A}}(x) = 1$, where sup is the supremum of a fuzzy set
   - The difference between the maximum and the supremum of a set: the maximum belongs to the set, the supremum may belong or not to that set
   - If a fuzzy set is not normal, it can be normalized by dividing its membership function by the supremum of the set, resulting the normalized fuzzy set:

$$\mu_{\tilde{A}_{norm}}(x) = \frac{\mu_{\tilde{A}}(x)}{\sup_x \mu_{\tilde{A}}(x)}$$

Properties of fuzzy sets: support, core, boundary

2. The support of a fuzzy set
   - The *support* of a fuzzy set (denoted *supp*) is the crisp set of all $x \in X$ for which $\mu_{\tilde{A}}(x) > 0$
   - In the example with the comfortable house it is the set $supp(\tilde{A}) = \{1, 2, 3, 4, 5, 6, 7\}$
   - Usually the elements of a fuzzy set having the degree of membership equal to 0 are not listed
3. The *(core)* of a fuzzy set:
   - is the crisp set for which $\mu_{\tilde{A}}(x) = 1$
4. The *(boundary)* of a fuzzy set:
   - is the crisp set for which $0 < \mu_{\tilde{A}}(x) < 1$

Exercise: represent graphically the support, the core and the boundary for a continuous trapezoidal fuzzy set.

Properties of a fuzzy set: α-level sets

5. The α-*level sets* (or α-*cuts*):
   - The α-*level set* (where $\alpha \in [0,1]$) of the fuzzy set $\tilde{A}$ having the membership function $\mu_{\tilde{A}}(x)$ is the crisp set A_α for which $\mu_{\tilde{A}}(x) \geq \alpha$
   - We can define *strong* α *cut* as the crisp set A'_α for which $\mu_{\tilde{A}}(x) > \alpha$
   - In the example with the comfortable house, WHERE $\tilde{A} = \{(1,0.1),(2,0.5),(3,0.8),(4,1.0),(5,0.7),(6,0.2)\}$, the α-cuts of the fuzzy set $\tilde{A}$ are:
     - $A_{0.1} = \{1,2,3,4,5,6\} = supp\tilde{A}$ (the support of $\tilde{A}$)
     - $A_{0.2} = \{2,3,4,5,6\}$
     - $A_{0.5} = \{2,3,4,5\}$
     - $A_{0.7} = \{3,4,5\}$
     - $A_{0.8} = \{3,4\}$
     - $A_{1.0} = \{4\} = \text{core}\tilde{A}$

Properties of a fuzzy set: α-level sets

- It can be proved that for any fuzzy set $\tilde{A}$, it holds:

$$\tilde{A} = \bigcup_{\alpha} \alpha \cdot A_{\alpha}$$

- Which means that, any fuzzy set can be written as the union for all the values of α of the product between α and the α-cuts of the fuzzy set
- This property is very important and it connects the fuzzy and the crisp sets
- It is also very useful for proving different properties of fuzzy sets (some properties are easier to be proved for crisp sets)

Properties of a fuzzy set: α-level sets

- We will illustrate this property on the example with the comfortable house:
  - $\alpha \cdot A_\alpha$ is the fuzzy set in which each element will hace the membership function equal with α.
  - $0.1 \cdot A_{0.1} = \{(1, 0.1), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0.1)\}$
  - $0.2 \cdot A_{0.2} = \{(2, 0.2), (3, 0.2), (4, 0.2), (5, 0.2), (6, 0.2)\}$
  - $\ldots$
  - $0.8 \cdot A_{0.8} = \{(3, 0.8), (4, 0.8)\}$
  - $1.0 \cdot A_{1.0} = \{(4, 1.0)\}$
  - The union of two or more fuzzy sets is defined as the maximum between their membership function, hence
  - $0.1 \cdot A_{0.1} \cup 0.2 \cdot A_{0.2} \cup \ldots \cup 0.8 \cdot A_{0.8} \cup 1.0 \cdot A_{1.0} =$ $= \{(1, 0.1), (2, max(0.1, 0.2)), (3, max(0.1, 0.2, \ldots, 0.8)),$ $(4, max(0.1, \ldots, 0.8, 1)), \ldots (6, max(0.1, 0.2)\} = \tilde{A}$

Properties of fuzzy sets: convexity

6. Convexity of a fuzzy set
   - A fuzzy set $\tilde{A} \subset X$ is convex if and only if $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0,1]$ the following relation takes place: $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \cdot x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$
   - The expression $\lambda \cdot x_1 + (1-\lambda) \cdot x_2$ describes the segment situated between the points having the abscissa x_1 and x_2
   - The expression $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \cdot x_2)$ describes the image of this segment through the function $\mu_{\tilde{A}}(x)$
   - **Equivalently, a fuzzy set $\tilde{A}$ is convex iff all its α-level sets are convex**
   - Which means that, if a fuzzy set is not convex, there exist α-level sets of this fuzzy set that are not convex, i.e., there exist segments $x_1^\alpha x_2^\alpha$ which are "interrupted" (are not continues)

Ex: Represent graphically a continuous and convex fuzzy set and a continuous non-convex fuzzy set.

Properties of fuzzy sets: cardinality

7. Cardinality of a fuzzy set
   - Cardinality of a finite fuzzy set $\tilde{A} \subset X$, denoted $|\tilde{A}|$ is defined as:
   $$|\tilde{A}| = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)$$
   - For a continuous fuzzy set $\tilde{A} \subset X$, its cardinality is defined:
   $$|\tilde{A}| = \int_{X} \mu_{\tilde{A}}(x) dx$$
   if the integral exist

7' Relative cardinality of a fuzzy set
   - Is denoted $||\tilde{A}||$
   - Is defined as $||\tilde{A}|| = \frac{|\tilde{A}|}{|X|}$, if it exists, where X is the universe of discourse for the set $\tilde{A}$

How to chose the membership functions

- Like in other aspects of the fuzzy sets theory, there are no clear "recipes" for choosing the membership functions of the fuzzy sets
- If we want to reduce the computations, we will prefer linear membership functions, i.e., triangles and trapeziums
- There are cases when we prefer non-linear membership functions (trigonometric, Gauss-type, etc):
  - There exist researchers that consider that linear membership functions do not provide the best results for some problems, while non-linear functions perform better
  - Sometimes the problem or the domain might need some types of membership functions
  - If we combine fuzzy sets theory with other methods, e.g., neural networks, it can be necessary to use membership functions that are suitable for these methods.

Outline

Operations for fuzzy sets: union, intersection, complement

- Given two fuzzy sets $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ and $\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) | x \in X\}$ over the same universe of discourse X, we can define operations of union, intersection and complement. We define:
- the *union* of the fuzzy sets $\tilde{A}$ si $\tilde{B}$ as the fuzzy set $\tilde{C} = \tilde{A} \cup \tilde{B}$, given by $\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) | x \in X\}$, where

$$\mu_{\tilde{C}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- the *intersection* of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ as the fuzzy set $\tilde{D} = \tilde{A} \cap \tilde{B}$, given by $\tilde{D} = \{(x, \mu_{\tilde{D}}(x)) | x \in X\}$, where

$$\mu_{\tilde{D}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- the *complement* of $\tilde{A}$ in X as the fuzzy set $\tilde{E} = \mathbb{C}_{\tilde{A}} X$ given by $\tilde{E} = \{(x, \mu_{\tilde{E}}(x)) | x \in X\}$, where

$$\mu_{\tilde{E}}(x) = 1 - \mu_{\tilde{A}}(x)$$

Operations with fuzzy sets: inclusion, equality

- *inclusion* of fuzzy sets: given two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X, the inclusion $\tilde{A} \subseteq \tilde{B}$ takes place iff $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $(\forall) x \in X$
- *equality* of two fuzzy sets: two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X are equals iff $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $(\forall) x \in X$
- Equivalently, two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X are equals iff $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$

Operations with fuzzy sets: examples

1. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "comfortable house for a 4 persons - family" and $\tilde{B} =$ "small house", where
$\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$ and
$\tilde{B} = \{(1, 1), (2, 0.8), (3, 0.4), (4, 0.1)\}$:
$\tilde{A} \cup \tilde{B} = \{(1, \max(0.1, 1)), (2, \max(0.5, 0.8)), (3, \max(0.8, 0.4)),$
$(4, \max(1, 0.1)), (5, \max(0.7, 0)), (6, \max(0.2, 0)\} =$
$\{(1, 1), (2, 0.8), (3, 0.8), (4, 1), (5, 0.7), (6, 0.2)\}$
$\tilde{A} \cap \tilde{B} = \{(1, \min(0.1, 1)), (2, \min(0.5, 0.8)), (3, \min(0.8, 0.4)),$
$(4, \min(1, 0.1)), (5, \min(0.7, 0)), (6, \min(0.2, 0)\} =$
$\{(1, 0.1), (2, 0.5), (3, 0.4), (4, 0.1), (5, 0), (6, 0)\}$
$\tilde{A} \cup \tilde{B}$ can be read as "comfortable house for a 4 persons - family or small", and $\tilde{A} \cap \tilde{B}$ as "comfortable house for a 4 persons - family and small"

Operations with fuzzy sets: examples (continued)

2. Determine $\mathbb{C}_{\tilde{A}}X$, where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$: ("non-comfortable house for a 4 persons - family") $\mathbb{C}_{\tilde{A}}X = \{(1, 1-0.1), (2, 1-0.5), (3, 1-0.8), (4, 1-1), (5, 1-0.7), (6, 1-0.2), (7, 1-0), (8, 1-0), (9, 1-0), (10, 1-0)\} = \{(1, 0.9), (2, 0.5), (3, 0.2), (4, 0), (5, 0.3), (6, 0.8), (7, 1), (8, 1), (9, 1), (10, 1)\}$
3. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "real numbers close to 10" and $\tilde{B} =$ "real number considerably larger than 11".
   - Analytically: $\tilde{C} = \tilde{A} \cup \tilde{B}$ si $\tilde{D} = \tilde{A} \cap \tilde{B}$, where $\mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$, $\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$ ...
   - Graphically: (more suited in this case), in the next slides:

Example of operations with fuzzy sets: union

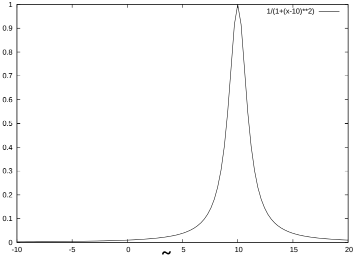

Figure 3: $\tilde{A}$ with $\mu_{\tilde{A}}(x)$

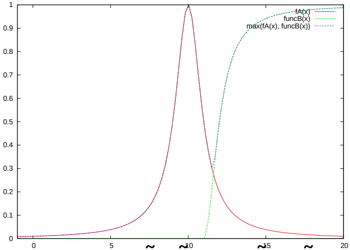

Figure 5: $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$

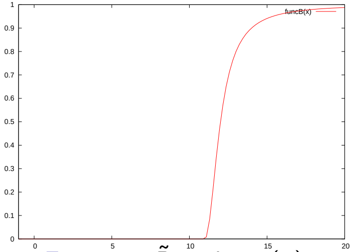

Figure 4: $\tilde{B}$ with $\mu_{\tilde{B}}(x)$

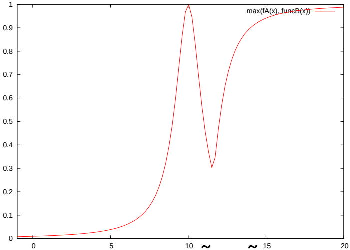

Figure 6: $\tilde{A} \cup \tilde{B}$

Example of operations with fuzzy sets: intersection

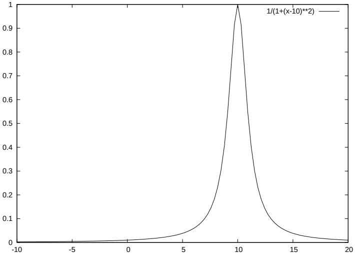

Figure 7: $\mu_{\tilde{A}}(x)$

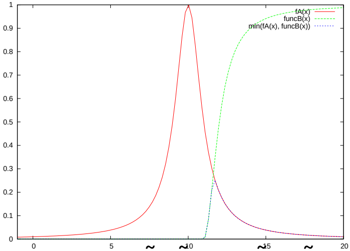

Figure 9: $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cap \tilde{B}$

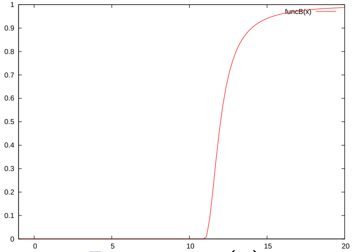

Figure 8: $\mu_{\tilde{B}}(x)$

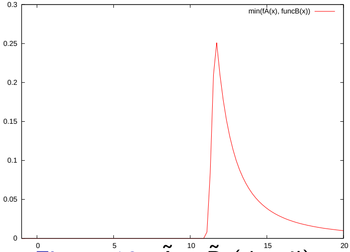

Figure 10: $\tilde{A} \cap \tilde{B}$ (detail)

Example of operations with fuzzy sets: exercises

1. Determine $\mathbb{C}_{\tilde{B}}X$, where $\tilde{B}$ is the fuzzy set "small house", and $X = \{1, 2, \dots, 9, 10\}$
2. Determine the complement of a fuzzy set that has a continuous trapezoidal-shaped membership function
3. For this fuzzy set, determine the union and intersection between the fuzzy set and its complement. What do you see ?

Outline

Properties of the operations with crisp sets and fuzzy sets

For crisp sets in the universe of discourse X the following properties are true (after [NR74]):

1. Commutativity:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
2. Associativity:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
3. Distributivity:
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
4. Idempotency:
$$A \cup A = A$$
$$A \cap A = A$$

Properties of the operations with crisp sets and fuzzy sets

5. Identity:

$$A \cup \emptyset = \emptyset \cup A = A$$

$$A \cup X = X \cup A = X$$

$$A \cap \emptyset = \emptyset \cap A = \emptyset$$

$$A \cap X = X \cap A = A$$

6. Transitivity: if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
7. Involution: $\overline{\overline{A}} = A$, where $\overline{A} = \mathbb{C}_A X$
8. De Morgan:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Properties of the operations with crisp sets and fuzzy sets

9. Absorption:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

10. Excluded middle laws *(excluded middle laws)*:

$$A \cup \overline{A} = X$$

$$A \cap \overline{A} = \emptyset$$

- Proprieties 1–9 hold for fuzzy sets, too, but NOT the property 10.
- Some researchers consider this fact (non-fulfillment of the excluded middle laws) as being the main characteristic of fuzzy sets.

Outline

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Extensions of fuzzy sets. Type m fuzzy sets

1. Type m fuzzy sets, with $m \geq 2$
   - Definition from previous section corresponds for type 1 fuzzy sets
   - **Definition:** a type 2 fuzzy set is defined as a fuzzy set whose membership values are a type 1 fuzzy set.
   - **Definition:** In a similar manner we can define a type m fuzzy set, with $m \geq 2$, as a fuzzy set whose membership values are a type $m - 1$ fuzzy sets.
   - In practice the fuzzy sets of type greater than 2 or 3 are hardly, if ever, used.
   - The operations with fuzzy sets of type 2 or greater can be defined only using the extension principle.
   - Even for type 2 fuzzy sets, the operations of union, intersection, complement, etc, imply many computations

Extensions of fuzzy sets

2. $\mathbb{L}$-fuzzy sets: fuzzy sets for which the real number interval $[0,1]$ is extended to a set $\mathbb{L}$ named partially ordered set (POS):
    - The interval $[0,1]$ is a POS
3. $\mathbb{B}$-fuzzy sets: similar with $\mathbb{L}$-fuzzy sets, but $\mathbb{B}$ is a o Boole algebra
4. Probabilistic sets (Hirota)
    - Are fuzzy sets with a membership function $\mu_{\tilde{A}}(x,\omega)$
    - For a certain, fixed x, the function becomes a random variable, that has a mean value and a variance
    - The mean value of a probabilistic fuzzy set is a regular fuzzy set
    - The combination between probability and fuzzy can be used, for example, for computing the reliability of some very complex systems (e.g. nuclear plants, in avionics, etc)
    - Fuzzy sets are better suited for modeling the human behaviour (e.g., the apparition of a human error in the operation of a complex system), while probabilities models better the failure of equipments

Extensions of fuzzy sets: IFS – Intuitionistic Fuzzy Sets

5. An intuitionistic fuzzy set (IFS) in univers of discourse X is defined as the triplet:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in X\}$$

$$\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) : X \to [0, 1], 0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$$

- $\mu_{\tilde{A}}(x)$ is named the degree of membership of x to $\tilde{A}$
- $\nu_{\tilde{A}}(x)$ is named the degree of non-membership of x to $\tilde{A}$
- For classic fuzzy sets it holds: $\nu_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$
- Similar with classic fuzzy sets, IFS can be extended to $\mathbb{L} - IFS$ or $\mathbb{B} - IFS$!!
- IFS have been proposed by Atanasov and Stoeva
- For IFS there can be defined operations of union, intersection and complement

Operations with IFS

We present the operations with IFS after [Ata86], [DBR01]

1. $\tilde{A} \subset \tilde{B}$ iff $(\forall) x \in X$ $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x)$
2. $\tilde{A} = \tilde{B}$ iff $\tilde{A} \subset \tilde{B}$ and $\tilde{B} \subset \tilde{A}$.
   Equivalently $\tilde{A} = \tilde{B}$ iff $(\forall) x \in X$ $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x)$
3. $\overline{\tilde{A}} = \{(x, \nu_{\tilde{A}}(x), \mu_{\tilde{A}}(x)) | x \in X\}$
4. $\tilde{A} \cup \tilde{B} = \{(x, \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}) | x \in X\}$
5. $\tilde{A} \cap \tilde{B} = \{(x, \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}) | x \in X\}$
6. $\tilde{A} + \tilde{B} =$
   $\{(x, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x), \nu_{\tilde{A}}(x) \cdot \nu_{\tilde{B}}(x)) | x \in X\}$
7. $\tilde{A} \cdot \tilde{B} = \{(x, \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x), \nu_{\tilde{A}}(x) + \nu_{\tilde{B}}(x) - \nu_{\tilde{A}}(x) \cdot \nu_{\tilde{B}}(x)) | x \in X\}$

Outline

Extensions of the operations with fuzzy sets

1. *Cartesian product:* Given the fuzzy sets $\tilde{A}_1, \ldots, \tilde{A}_n$ in the universes $X_1, \ldots, X_n$, the cartesian product of these fuzzy sets is a fuzzy set the product space $X_1 \times \ldots \times X_n$ with the membership function

$$\mu_{\tilde{A}_1 \times \ldots \tilde{A}_n}(x) = min_i[\mu_{\tilde{A}_i}(x_i)]$$

, where $x = (x_1, \ldots, x_n)$, $x_i \in X_i$

   - Example of cartesian product for $n = 2$

2. *The m-th power of a fuzzy set $\tilde{A}$:* in the universe of discourse X is the fuzzy set with the membership function

$$\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m$$

, where $x \in X$

3. Intersection and union operations with fuzzy sets are extended by the triangular, or *t-norms* (for intersection) and *s-norms*, named also *t-co-norms* (the union).

t-norms: definitions

Definition

A *t-norm* is a two valued function $t : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

1. $t(0,0) = 0$,
   $t(1, \mu_{\tilde{A}}(x)) = t(\mu_{\tilde{A}}(x), 1) = \mu_{\tilde{A}}(x)$, $(\forall)\ x \in X$, i.e.
   $(\forall)\ \mu_{\tilde{A}}(x) \in [0,1]$
2. monotony:
   $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
3. commutativity:
   $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$
4. associativity:
   $t(\mu_{\tilde{A}}(x), t(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$

s-norms: definitions

Definition

An *s-norm (t-conorm)* is a two valued function $s : [0,1] \times [0,1] \to [0,1]$ following properties:

1. $s(1,1) = 1$,
   $s(0, \mu_{\tilde{A}}(x)) = s(\mu_{\tilde{A}}(x), 0) = \mu_{\tilde{A}}(x)$, $(\forall)\ x \in X$, i.e.
   $(\forall)\ \mu_{\tilde{A}}(x) \in [0,1]$
2. monotony:
   $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq s(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
3. commutativity:
   $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = s(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$
4. associativity:
   $s(\mu_{\tilde{A}}(x), s(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = s(s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$

s-norms and t-norms

- The t-norms and s-norms obeys De Morgan's law, i.e. $A \cap B = \overline{\overline{A} \cup \overline{B}}$:
- $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = 1 - s(1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x))$
- Based on this relation, we can generate a t-norm starting from an s-norm and the opposite, we can generate an s-norm starting from its corresponding t-norm.
- Next, we will give some examples of t-norm, s-norm pairs.

Pairs of t-norms and s-norms

1. Drastic product, drastic sum:

$$t_w(\mu_1(x), \mu_2(x)) = \begin{cases} \mu_1(x) \text{ if } \mu_2(x) = 1 \\ \mu_2(x) \text{ if } \mu_1(x) = 1 \\ 0, \text{ if } 0 \le \mu_1(x), \mu_2(x) < 1 \end{cases}$$

$$s_w(\mu_1(x), \mu_2(x)) = \begin{cases} \mu_1(x) \text{ if } \mu_2(x) = 0 \\ \mu_2(x) \text{ if } \mu_1(x) = 0 \\ 1, \text{ if } 0 < \mu_1(x), \mu_2(x) \le 1 \end{cases}$$

Equivalently,

$$t_w(\mu_1, \mu_2) = \begin{cases} \min(\mu_1, \mu_2) \text{ if } \max(\mu_1, \mu_2) = 1 \\ 0, \text{ otherwise} \end{cases}$$

$$s_w(\mu_1, \mu_2) = \begin{cases} \max(\mu_1, \mu_2) \text{ if } \min(\mu_1, \mu_2) = 0 \\ 1, \text{ otherwise} \end{cases}$$

Pairs of t-norms and s-norms

2. Bounded difference, bounded sum:
   $t_1(\mu_1(x), \mu_2(x)) = \max\{0, \mu_1(x) + \mu_2(x) - 1\}$
   $s_1(\mu_1(x), \mu_2(x)) = \min\{1, \mu_1(x) + \mu_2(x)\}$
3. Einstein product, Einstein sum:

$$t_{1.5}(\mu_1(x), \mu_2(x)) = \frac{\mu_1(x) \cdot \mu_2(x)}{2 - [\mu_1(x) + \mu_2(x) - \mu_1(x) \cdot \mu_2(x)]}$$

$$s_{1.5}(\mu_1(x), \mu_2(x)) = \frac{\mu_1(x) + \mu_2(x)}{1 + \mu_1(x) \cdot \mu_2(x)}$$

4. Algebraic product and sum:
   $t_2(\mu_1(x), \mu_2(x)) = \mu_1(x) \cdot \mu_2(x)$
   $s_2(\mu_1(x), \mu_2(x)) = \mu_1(x) + \mu_2(x) - \mu_1(x) \cdot \mu_2(x)$

Pairs of t-norms and s-norms

5. Hamacher product and sum:

$$t_{2.5}(\mu_1(x), \mu_2(x)) = \begin{cases} \frac{\mu_1(x)\cdot\mu_2(x)}{\mu_1(x)+\mu_2(x)-\mu_1(x)\cdot\mu_2(x)}, & \text{if } \mu_1, \mu_2 \neq 0 \\ 0, & \text{if } \mu_1 = \mu_2 = 0 \end{cases}$$

$$s_{2.5}(\mu_1(x), \mu_2(x)) = \begin{cases} \frac{\mu_1(x)+\mu_2(x)-2\cdot\mu_1(x)\cdot\mu_2(x)}{1-\mu_1(x)\cdot\mu_2(x)}, & \text{if } \mu_1, \mu_2 \neq 1 \\ 1, & \text{if } \mu_1 = \mu_2 = 1 \end{cases}$$

6. minimum and maximum:
   $t_3(\mu_1(x), \mu_2(x)) = \min\{\mu_1(x), \mu_2(x)\}$
   $s_3(\mu_1(x), \mu_2(x)) = \max\{\mu_1(x), \mu_2(x)\}$

t-norms and s-norms

The operators presented before satisfy the following relations:

$$t_w \leq t_1 \leq t_{1.5} \leq t_2 \leq t_{2.5} \leq t_3$$

$$s_3 \leq s_{2.5} \leq s_2 \leq s_{1.5} \leq s_1 \leq s_w$$

More general, Dubois and Prade have shown that each t-norms is bounded between the drastic product and by minimum, and that any s-norm is situated between the maximum and the drastic sum.

Which means, for all t-norm t and for all s-norm s are satisfied the following relations:

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \ x \in X$$

$$\max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \leq s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq s_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \ x \in X$$

Parametrized intersection and union operators: proposed by Hamacher

- Hamacher proposed the following operators for intersection and union:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{\gamma + (1-\gamma)\cdot(\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x))},\ \gamma \geq 0$$

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \frac{(\gamma'-1)\cdot\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x) + \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)}{1+\gamma'\cdot\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)},\ \gamma' \geq -1$$

- For $\gamma = 0$ and $\gamma' = -1$ we obtain Hamacher product and sum, and for $\gamma = 1$ and $\gamma' = 0$ we obtain algebraic product and sum.

Parametrized intersection and union operators: proposed by Yager

- Yager proposed the following operators for intersection and union:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = 1 - \min\{1, [(1-\mu_{\tilde{A}}(x))^p + (1-\mu_{\tilde{B}}(x))^p]^{1/p}\},\ p \geq 1$$

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \min\{1, [(\mu_{\tilde{A}}(x))^p + (\mu_{\tilde{B}}(x))^p]^{1/p}\},\ p \geq 1$$

- For $p = 1$ we obtain bounded difference and sum, while for $p \to \infty$, Yager's operators converge to *minimum* and respectively *maximum*
- Yager's operators satisfy De Morgan laws, are commutative and associative for all p, monotone non-decreasing in $\mu(x)$ and include the classic operators from dual (clasic) logic
- However, Yager's operators are not distributive

Outline

Criteria for selecting the operators

1. *Axiomatic strength:* if all other characteristics are equal, is better the operator which satisfies the less restrictive (limiting) set of axioms
2. *Empirical fit:* it is important that the chosen operators to be appropriate for the domain where they are used. This can be verified only through empiric tests .
3. *Adaptability:* if one wants to have a small number of operators to model many situations, then it is recommended to use parametrized operators (Yager, Hamacher). Of course, these operators are not numerically efficient (they require complex computations)
4. *Computational (numerical) efficiency:* it is obvious that *max* and *min* operators imply less computational effort than parametrized operators, hence, they will be preferred if we want a good numerical efficiency.

Criteria for selecting the operators

5. *Compensation:* Compensation is defined:
   given $k \in [0,1]$, if $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = k$, operator t is compensatory if, modifying $\mu_{\tilde{A}}(x)$, it can be obtained $t(\mu_{\tilde{A}}(x_k), \mu_{\tilde{B}}(x_k)) = k$ for another $\mu_{\tilde{B}}(x)$.

Example:
If $\mu_{\tilde{A}} = 0.2$ and $\mu_{\tilde{B}} = 0.3$, if the operator t is min, we obtain $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min(0.2, 0.3) = 0.2$ (hence $k = 0.2$)
If we make $\mu_{\tilde{A}} = 0.1$, then, no matter what would be the value of $\mu_{\tilde{B}}(x)$, we **cannot** obtain $\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = 0.2$ because $\min(0.1, \mu_{\tilde{B}}(x)) \leq 0.1$, hence the operator min is not compensatory.

If the operator t is the algebraic product, then we obtain: $0.2 \cdot 0.3 = 0.06$, hence $k = 0.06$ and if we make $\mu_{\tilde{A}}(x) = 0.1$, we have $0.1 \cdot \mu_{\tilde{B}}(x) = 0.06 \implies \mu_{\tilde{B}}(x) = \frac{0.06}{0.1} = 0.6$, hence the operator algebraic product is compensatory.

Criteria for selecting the operators

- In general, if we want to solve a problem where the time constrains are important, then the numerical efficiency criterion is of prime importance, because we want to obtain the results in real time. This is the case, e.g., in control engineering
- If, on the other side, we solve a complex problem, e.g., an expert system for medical diagnosis, then the complexity and the “finesse” of the operator is more important than to obtain quickly the result.
- Beside the above criteria, we can consider other criteria as well, e.g., *technological fit*: if the fuzzy operations are implemented in hardware, then some operators can be easier implemented in certain technologies.

Outline

Outline

The importance of the extension principle

- In its general form has been formulated by Zadeh in 1973
- It is very important because it can be used to extend different mathematical and non-mathematical theories or domains through the framework of fuzzy logic (to be "fuzzified")
- Hence, there exist fuzzy numbers, fuzzy arithmetic, fuzzy analysis, fuzzy probabilities. but also fuzzy codes, fuzzy automata, fuzzy flip-flops, etc

Functions

- We recall that a function f is an application $f : X \rightarrow Y$ that associates to each $x \in X$ an unique $y \in Y$.
- That is, $(\forall)x \in X$ $(\exists)$ an unique $y \in Y$ such that $y = f(x)$, named the image of x through the function f.
- This means that:
  1. No $x \in X$ has two or more images, e.g. y_1 si y_2
  2. No x from X has 0 (zero) images through function f.
- FIGURE !!!!

Function

- A function is *injective*, if $(\forall)\ x_1, x_2 \in X$ with $x_1 \neq x_2$, it holds $f(x_1) \neq f(x_2)$ (every y from Y is the image of at most one x from X)
- A function is *surjective*, if $(\forall) y \in Y\ (\exists) x \in X$ such that $y = f(x)$ (every y from Y is the image of at least one x from X)
- A function is *bijective* iff it is both injective and surjective (every y from Y is the image of *exactly one* x from X)
- In other words, to every x in X it corresponds an *unique* y in Y, or, between the sets X and Y it can be established a "one to one" correspondence.
- A function is invertible iff it is bijective.
- The Inverse of a function f is denoted cu f^{-1}: $f^{-1} : Y \to X$ such that $x = f^{-1}(y)$
- In the following, we will make an abuse of notation, i.e., we will denote $x = f^{-1}(y)$ even if the function f is not invertible.

Outline

The extension principle in its reduced form

Definition: Let X, Y, be universes, $\tilde{A} \subset X$ be a fuzzy set in X and let $f : X \to Y$ be a function (a mapping) such that $y = f(x)$. The extension principle allows the definition of a fuzzy set $\tilde{B} \subset Y$, $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x),\ x \in X\}$, where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } (\exists)\ f^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

Explanations

If we have a fuzzy set $\tilde{A} \subset X$ and a function $f : X \to Y$, the extension principle allow us to determine the fuzzy set $\tilde{B} \subset Y$ which is the image (the mapping) of the set $\tilde{A}$ through the function f. The following situations may appear:

- If an element $y \in Y$ is the image of a *unique* element $x \in X$, then it is straight forward to consider $\mu_{\tilde{B}}(y) = \mu_{\tilde{A}}(x)$
- If an element $y \in Y$ is the image of *no one* element $x \in X$, then it would be normal to consider that $\mu_{\tilde{B}}(y) = 0$
- If an element $y \in Y$ is the image of several elements $x \in X$ (for example $x_i, x_j, \ldots, x_k$), then, the degree of membership of y la $\tilde{B}$ will be the maximum between the degrees of membership of the elements $x_i, x_j, \ldots, x_k$ to $\tilde{A}$

In formula, by $x \in f^{-1}(y)$ are denoted those elements x form X whose image through function f is y from Y, i.e. $y = f(x)$.

The extension principle in its general form

Definition: Let $X = X_1 \times X_2 \times \ldots \times X_n$ be the cartesian product of the universes X_i, $i = 1, \ldots, n$, and $\tilde{A}_i \subset X_i$ be fuzzy sets in X_i, and let $f : X \to Y$ be a function (a mapping) such that $y = f(x_1, x_2, \ldots, x_n)$, $x_i \in X_i$, $i = 1, \ldots, n$, and Y is also an universe.

The extension principle permits the definition of a fuzzy set $\tilde{B} \subset Y$, $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2, \ldots, x_n),\ (x_1, x_2, \ldots, x_n) \in X\}$, where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} min(\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)), & \text{if } (\exists)\ f^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

where $x = (x_1, x_2, \ldots, x_n)$

Explanations

- In the general form, the universe X is the cartesian product of the universes $X_1, X_2, \ldots X_n$, and the fuzzy set $\tilde{A}$ is the cartesian product of the fuzzy sets $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$, i.e. $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2 \times \ldots \times \tilde{A}_n$
- According to the formula for the cartesian product, it results that $\mu_{\tilde{A}}(x) = \min(\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n))$, where $x = (x_1, x_2, \ldots, x_n)$
- Of course, the extension principle in the reduced form can be obtained from the extension principle in the general form, by making $n = 1$.

Outline

Applications of the extension principle in the reduced form

Let $X = Y = \mathbb{Z}$, the set of integer numbers, let $\tilde{A} \subset \mathbb{Z}$, $\tilde{A} = \{(-2, 0.3), (-1, 0.5), (0, 0.8), (1, 1), (2, 0.7), (3, 0.1)\}$ and let the function $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$. Which is the fuzzy set $\tilde{B} \subset \mathbb{Z}$ with $\tilde{B} = f(\tilde{A})$ (i.e. $\tilde{B}$ is the image of the set $\tilde{A}$ through the function f) ?.

- $\mu_{\tilde{B}}(0) = \mu_{\tilde{A}}(0) = 0.8$ and $\mu_{\tilde{B}}(9) = \mu_{\tilde{A}}(3) = 0.1$
- $\mu_{\tilde{B}}(1) = \sup_{x \in f^{-1}(1)}(\mu_{\tilde{A}}(x)) = \sup(\mu_{\tilde{A}}(-1), \mu_{\tilde{A}}(1))$
  $= \sup(0.5, 1) = 1$
- $\mu_{\tilde{B}}(4) = \sup_{x \in f^{-1}(4)}(\mu_{\tilde{A}}(x)) = \sup(\mu_{\tilde{A}}(-2), \mu_{\tilde{A}}(2))$
  $= \sup(0.3, 0.7) = 0.7$
- If we consider that all the integer numbers between 0 and 9belong to the set $\tilde{B}$, then: $\mu_{\tilde{B}}(2) = \mu_{\tilde{B}}(3)$ $= \mu_{\tilde{B}}(5) = \mu_{\tilde{B}}(6) = \mu_{\tilde{B}}(7) = \mu_{\tilde{B}}(8) = 0$ because $2, 3, 5, 6, 7$ si 8 are not the image of any element from $\tilde{A}$ through the function f.
- In conclusion, $\tilde{B} = \{(0, 0.8)(1, 1), (2, 0), (3, 0), (4, 0.7), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0.1)\}$

Applications of the extension principle: the addition of two discrete fuzzy numbers

Given $X_1 = X_2 = Y = \mathbb{Z}$ and the fuzzy sets $\tilde{A}_1 \subset X_1$ *approximately 2* and $\tilde{A}_2 \subset X_2$ *approximately 6* described as:
$\tilde{A}_1 = \{(1, 0.2), (2, 1), (3, 0.5), (4, 0.1)\}$,
$\tilde{A}_2 = \{(5, 0.2), (6, 1), (7, 0.5), (8, 0.1)\}$.
We want to obtain the fuzzy set $\tilde{B} \subset Y$ given by $\tilde{B} = \tilde{A}_1 \oplus \tilde{A}_2$, where the symbol $\oplus$ represents the addition of the fuzzy numbers, defined as follows:

$$\mu_{\tilde{B}}(y) = \sup_{y = x_1 + x_2} (\min(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2)))$$

Answer:
$\tilde{B} = \{(6, 0.2), (7, 0.2), (8, 1), (9, 0.5), (10, 0.5), (11, 0.1), (12, 0.1)\}$

- We notice that we obtain the number *approximately 8*, as expected, but the "width" of the sum (i.e., the degree of imprecision) is bigger than the degree of imprecsion of the terms (the added fuzzy numbers).

Addition of two discrete fuzzy numbers

Example: computation of $\mu_{\tilde{B}}(9)$:

- We start from $y = x_1 + x_2$ which, for $y = 9$, becomes:
  - $9 = 1 + 8$
  - $9 = 2 + 7$
  - $9 = 3 + 6$
  - $9 = 4 + 5$
- Replacing in formula $\mu_{\tilde{B}}(y) = \sup_{y=x_1+x_2}\{\min(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2))\}$ we obtaine:
- $\mu_{\tilde{B}}(9) = \sup_{9=x_1+x_2}\{\min(\mu_{\tilde{A}_1}(1), \mu_{\tilde{A}_2}(8))), \min(\mu_{\tilde{A}_1}(2), \mu_{\tilde{A}_2}(7))), \min(\mu_{\tilde{A}_1}(3), \mu_{\tilde{A}_2}(6))), \min(\mu_{\tilde{A}_1}(4), \mu_{\tilde{A}_2}(5)))\} = \sup\{\min(0.2, 0.1), \min(1, 0.5), \min(0.5, 1), \min(0.1, 0.2)\} = \sup(0.1, 0.5, 0.5, 0.1) = 0.5$

Conclusions: directions in fuzzy logic

1. The direction followed by mathematicians, who aim to:
   - on the one side, to give a theoretical foundation to the results, operators and formulas from fuzzy logic
   - on the other side, try to extend other domains, mathematical or non-mathematical, through the framework of fuzzy logic.
   - Hence, there exists fuzzy numbers, fuzzy arithmetic, fuzzy functions, fuzzy calculus, fuzzy probabilities, but also fuzzy automata, fuzzy flip-flops, fuzzy codes, fuzzy reliability, etc
2. The second direction is followed by engineers, economists, linguists, medical doctors, etc, who apply the results of fuzzy logic in their domains of activity
   - They must keep themselves informed on the results obtained by mathematicians.

Outline

Outline

Definitions

- In this course we will discuss only binary fuzzy relations
- The generalization to n-ary fuzzy relations is straightforward
- Fuzzy relations are subsets of the cartesian product $X \times Y$, where X and Y are universes of discourse.

Definition

Given the universes of discourse X and Y, a fuzzy relation $\tilde{R}$ in $X \times Y$ is defined as the set
$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$, where
$\mu_{\tilde{R}}(x, y) : X \times Y \to [0, 1]$

Examples of fuzzy relations

1. For $X = Y = \mathbb{R}$, we define the continuous fuzzy relation "x considerably larger than y" :

$$\mu_{\tilde{R}}(x,y) = \begin{cases} 0, \text{ if } x \leq y \\ \frac{|x-y|}{10 \cdot |y|}, \text{ if } y < x \leq 11 \cdot y \\ 1, \text{ if } x > 11 \cdot y \end{cases}$$

2. The fuzzy relation "$x \gg y$" could be defined also as:

$$\mu_{\tilde{R}}(x,y) = \begin{cases} 0, \text{ if } x \leq y \\ \frac{(x-y)^2}{1+(x-y)^2}, \text{ if } x > y \end{cases}$$

Examples of fuzzy relations

3. For the discrete fuzzy sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, the fuzzy relation $\tilde{R}$ "$x \gg y$" can be expressed by the matrix:

	y_1	y_2	y_3
x_1	0.5	1	0
x_2	0.7	0.2	0.1

Table 1: Fuzzy relation $\tilde{R}$

Definition

Definition

Let $X, Y \subseteq \mathbb{R}$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y \in Y\}$$

Then $\tilde{R} = \{(x, y, \mu_{\tilde{R}}(x, y)) | (x, y) \in X \times Y\}$ is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{A}}(x), \ \forall (x, y) \in X \times Y$$

si

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{B}}(y), \ \forall (x, y) \in X \times Y$$

.

This definition is useful for fuzzy graphs, but we will not study fuzzy graphs in this course.

Union and intersection of fuzzy relations

Being fuzzy sets, between fuzzy relations defined on the same universes of discourse we can perform union and intersection operations as follows:

Definition

Let $\tilde{R}$ and $\tilde{Z}$ two fuzzy relations defined in the same product space. Union, respectively intersection of the fuzzy relations $\tilde{R}$ and $\tilde{Z}$ are defined:

$$\mu_{\tilde{R}\cup\tilde{Z}}(x,y) = \max\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y)\}, \quad (x,y) \in X \times Y$$

$$\mu_{\tilde{R}\cap\tilde{Z}}(x,y) = \min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y)\}, \quad (x,y) \in X \times Y$$

Outline

Composition of fuzzy relations

- Discrete fuzzy relations, described by matrixes, can be composed in a similar way with matrixes multiplication
- Like in the case of matrixes, in order to compose the fuzzy relations, their dimensions have to match
- The most used composition method is the $\max - \min$ composition
- Given two fuzzy relations expressed in matrix form, after the *max-min* composition it results a new fuzzy relation (a matrix) whose elements are obtained by "multiplication" of the elements of the two relations
- "Multiplication" is made between a line from the first matrix and a column from the second matrix, but instead of $+$ and $\cdot$ we use max and min
- There exists other composition methods than *max-min*, more precisely the *min* operation can be replaced by algebraic product, average, or other operations.
- Composition of fuzzy relations is important for understanding fuzzy inference.

The max-min composition

Definition

Given the fuzzy relations $\tilde{R}_1(x, y) \subset X \times Y$ and $\tilde{R}_2(y, z) \subset Y \times Z$, their max-min composition, $\tilde{R}_1$ *max-min* $\tilde{R}_2$, denoted $\tilde{R}_1 \circ \tilde{R}_2$ is defined as the fuzzy set:

$$\tilde{R}_1 \circ \tilde{R}_2 = \{((x, z), \mu_{\tilde{R}_1 \circ \tilde{R}_2}(x, z)) \mid (x, z) \in X \times Z\}$$

, where

$$\mu_{\tilde{R}_1 \circ \tilde{R}_2}(x, z) = \max_y [\min(\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z))]$$

The max-star composition

Definition

Given the fuzzy relations $\tilde{R}_1(x,y) \subset X \times Y$ and $\tilde{R}_2(y,z) \subset Y \times Z$, their max-star composition, $\tilde{R}_1 \circledast \tilde{R}_2$ is defined as the fuzzy set:

$$\tilde{R}_1 \circledast \tilde{R}_2 = \{((x,z), \mu_{\tilde{R}_1 \circledast \tilde{R}_2}(x,z)) \mid (x,z) \in X \times Z\}$$

, where

$$\mu_{\tilde{R}_1 \circledast \tilde{R}_2}(x,z) = \max_y[(\mu_{\tilde{R}_1}(x,y) * \mu_{\tilde{R}_2}(y,z))]$$

If the operation $*$ (*star*) is associative and monotonically nondecreasing in each argument, then the $\max - *$ composition has similar properties with the $\max - \min$ composition.
The most employed are the *max-prod* $(\max - \cdot)$(when the operation $*$ is the algebraic product, and *max-average*, when operation $*$ is the arithmetic mean.

Examples of compositions of discrete binary fuzzy relations

Let $\tilde{R}_1(x,y)$ and $\tilde{R}_2(y,z)$ discrete binary fuzzy relations defined by the following matrices:

$\tilde{R}_1$:

	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.2	0	1	0.7
x_2	0.3	0.5	0	0.2	1
x_3	0.8	0	1	0.4	0.3

$\tilde{R}_2$:

	z_1	z_2	z_3	z_4
y_1	0.9	0	0.3	0.4
y_2	0.2	0.1	0.8	0
y_3	0.8	0	0.7	1
y_4	0.4	0.2	0.3	0
x_5	0	1	0	0.8

Examples of compositions of discrete binary fuzzy relations

The results of max-min, max-prod and max-average composition of the relations $\tilde{R}_1(x,y)$ and $\tilde{R}_2(y,z)$ are given by the following matrices:

$\tilde{R}_1 \ \max - \min \ \tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.7
x_2	0.3	1	0.5	0.8
x_3	0.8	0.3	0.7	1

$\tilde{R}_1 \ \max - \cdot \ \tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.56
x_2	0.27	1	0.4	0.8
x_3	0.8	0.3	0.7	1

Examples of compositions of discrete binary fuzzy relations

$\tilde{R}_1$ *max-average* $\tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.7	0.85	0.65	0.75
x_2	0.6	1	0.65	0.9
x_3	0.9	0.65	0.85	1

Outline

Properties of max-min composition

1. *Associativity* The *max-min* composition is associative, that is

$$(\tilde{R}_3 \circ \tilde{R}_2) \circ \tilde{R}_1 = \tilde{R}_3 \circ (\tilde{R}_2 \circ \tilde{R}_1)$$

   Consequence: the power of a fuzzy relation. If
   $\tilde{R}_1 = \tilde{R}_2 = \tilde{R}_3 = \tilde{R}$, we can write:
   $\tilde{R}^2 = \tilde{R} \circ \tilde{R}$,
   $\tilde{R}^3 = \tilde{R} \circ \tilde{R} \circ \tilde{R}$ and so on.
   Hence, for any n natural, we can define $\tilde{R}^n$

2. *Reflexivity Definition:* A fuzzy relation defined in $X \times X$ is reflexive iff (if and only if) $\mu_{\tilde{R}}(x, x) = 1$, $\forall x \in X$
   Property: If the fuzzy relations $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive, then their max-min composition, $\tilde{R}_1 \circ \tilde{R}_2$ is also reflexive.

Properties of max-min composition

3. *Symmetry Definition:* A fuzzy relation is called symmetric if $\tilde{R}(x, y) = \tilde{R}(y, x)$
4. *Antisymmetry Definition:* A fuzzy relation is called antisymmetric if, $\forall x, y \in X$, if $x \neq y$ then either $\mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x)$, or $\mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(y, x) = 0$
5. A fuzzy is called perfect antisymmetric if $\forall\ x \neq y$, whenever $\mu_{\tilde{R}}(x, y) > 0$, then $\mu_{\tilde{R}}(y, x) = 0$
6. *Transitivity Definition:* A fuzzy relation $\tilde{R}$ is called max-min transitive if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$.

Similarity, de pre-order and order fuzzy relations

Definition

A *similarity relation* is a fuzzy relation that is reflexive, symmetrical and max-min transitive

The idea of similarity is analogous to the idea of equivalence, being possible to create similarity trees.

Definition

A fuzzy relation which is max-min transitive and reflexive is called *fuzzy preorder relation*.

Definition

A fuzzy relation which is max-min transitive, reflexive and antisymmetric is called *fuzzy order relation*.

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