Fuzzy sets. Operations with fuzzy sets

Chapter 2

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Outline

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Crisp (classic) sets, fuzzy sets

- Given an *universe of discourse* (*crisp*) X
- For a classic (crisp) set $A \subset X$, for each element $x \in X$, either $x \in A$ or $x \notin A$.
- For the set A it can be defined a characteristic function $\nu_A : X \to \{0, 1\}$, with $\nu_A(x) = 1$ iff (if and only if) $x \in A$ and $\nu_A(x) = 0$ iff $x \notin A$
- For a fuzzy set $\tilde{A}$, an element $x \in X$ belongs to the fuzzy set $\tilde{A} \subset X$ **in a certain degree**
- The characteristic function of a crisp set will be extended to the *membership function* of a fuzzy set, which can take values in the real numbers interval $[0, 1]$

Definitions

Definition

"If X is a collection of objects" (named the *universe of discourse*) "denoted generically by x, then a fuzzy set $\tilde{A} \subset X$ is a set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

where $\mu_{\tilde{A}}(x) : X \to [0, 1]$ is called *membership function* or *degree of membership* (also, degree of compatibility or degree of truth). of x in A" (Zimmermann [Zim91])

If the interval of real numbers [0, 1] is replaced with the discrete set $\{0, 1\}$, then the fuzzy set $\tilde{A}$ becomes a classic (crisp) set.

Fuzzy sets. Examples of fuzzy sets

- Fuzzy sets can be discrete or continuous
- The interval $[0, 1]$ can be extended to $[0, k]$, where $k > 0$
- It is possible to define fuzzy sets on more complex structures than intervals or real numbers, e.g. $\mathbb{L}$-*fuzzy sets*, where $\mathbb{L}$ is a partially ordered set (see chapter 3, Extensions of fuzzy sets)
- Example of discrete fuzzy set (Zimmermann [Zim91]):
  - MF: comfortable house for a 4 person family as a function of the number of bedrooms:
  - The universe discourse: $X = \{1, 2, \ldots, 10\}$
  - $\tilde{A} \subset X$ will be $\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$

Examples of fuzzy sets (cnt'd)

Example of continuous fuzzy set: real numbers close to 10

- $X = \mathbb{R}$ (the set of real numbers)
- The membership function of the fuzzy set $\tilde{A} \subset \mathbb{R}$:

$$\mu_{\tilde{A}}(x) = \frac{1}{1 + (x - 10)^2} \tag{1}$$

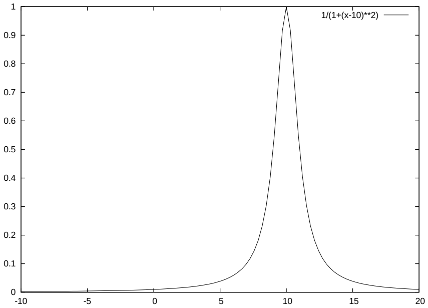

Figure 1: $\tilde{A}$ with $\mu_{\tilde{A}}(x) = \frac{1}{1+(x-10)^2}$

Examples of fuzzy sets (cnt'd)

Example of a continuous fuzzy set: real numbers considerably larger than 11

- $X = \mathbb{R}$ (the set of real numbers)
- The membership function of the fuzzy set: $\tilde{B} \subset \mathbb{R}$:

$$\mu_{\tilde{B}}(x) = \begin{cases} \frac{(x-11)^2}{1+(x-11)^2} & \text{if } x \geq 11 \\ 0, & \text{if } x < 11 \end{cases} \tag{2}$$

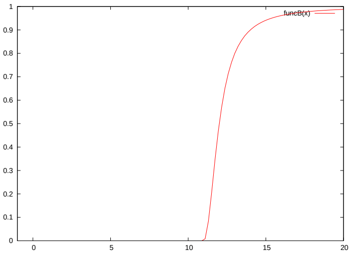

Figure 2: $\tilde{B}$ with $\mu_{\tilde{B}}(x)$

Notations for fuzzy sets

1. Pairs *(element, value)* for discrete fuzzy sets (like in the example with the comfortable house), respectively *(generic element, membership function)* for continuous fuzzy sets: e.g. $(x, \mu_{\tilde{A}}(x))$
2. Solely by stating the membership function (for continuous fuzzy sets)
3. As a "sum" for discrete fuzzy sets, respectively "integral" for continuous fuzzy sets (this notation may create confusions !!):

$$\tilde{A} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i} = \frac{\mu_{\tilde{A}}(x_1)}{x_1} + \frac{\mu_{\tilde{A}}(x_2)}{x_2} + \ldots + \frac{\mu_{\tilde{A}}(x_n)}{x_n}$$

$$\tilde{A} = \int \frac{\mu_{\tilde{A}}(x)}{x}$$

Caution, there are neither sums nor integrals here, these are only notations !!!

Outline

Properties (characteristics) of fuzzy sets: normal fuzzy sets

1. Normal fuzzy sets
   - A fuzzy set is called *normal* if $\sup_x \mu_{\tilde{A}}(x) = 1$, where sup is the supremum of a fuzzy set
   - The difference between the maximum and the supremum of a set: the maximum belongs to the set, the supremum may belong or not to that set
   - If a fuzzy set is not normal, it can be normalized by dividing its membership function by the supremum of the set, resulting the normalized fuzzy set:

$$\mu_{\tilde{A}_{norm}}(x) = \frac{\mu_{\tilde{A}}(x)}{\sup_x \mu_{\tilde{A}}(x)}$$

Properties of fuzzy sets: support, core, boundary

2. The support of a fuzzy set
   - The *support* of a fuzzy set (denoted *supp*) is the crisp set of all $x \in X$ for which $\mu_{\tilde{A}}(x) > 0$
   - In the example with the comfortable house it is the set $supp(\tilde{A}) = \{1, 2, 3, 4, 5, 6, 7\}$
   - Usually the elements of a fuzzy set having the degree of membership equal to 0 are not listed
3. The *(core)* of a fuzzy set:
   - is the crisp set for which $\mu_{\tilde{A}}(x) = 1$
4. The *(boundary)* of a fuzzy set:
   - is the crisp set for which $0 < \mu_{\tilde{A}}(x) < 1$

Exercise: represent graphically the support, the core and the boundary for a continuous trapezoidal fuzzy set.

Properties of a fuzzy set: α-level sets

5. The α-*level sets* (or α-*cuts*):
   - The α-*level set* (where $\alpha \in [0, 1]$) of the fuzzy set $\tilde{A}$ having the membership function $\mu_{\tilde{A}}(x)$ is the crisp set A_α for which $\mu_{\tilde{A}}(x) \geq \alpha$
   - We can define *strong* α *cut* as the crisp set A'_α for which $\mu_{\tilde{A}}(x) > \alpha$
   - In the example with the comfortable house, WHERE $\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$, the α-cuts of the fuzzy set $\tilde{A}$ are:
     - $A_{0.1} = \{1, 2, 3, 4, 5, 6\} = supp\tilde{A}$ (the support of $\tilde{A}$)
     - $A_{0.2} = \{2, 3, 4, 5, 6\}$
     - $A_{0.5} = \{2, 3, 4, 5\}$
     - $A_{0.7} = \{3, 4, 5\}$
     - $A_{0.8} = \{3, 4\}$
     - $A_{1.0} = \{4\} = \text{core}\tilde{A}$

Properties of a fuzzy set: α-level sets

- It can be proved that for any fuzzy set $\tilde{A}$, it holds:

$$\tilde{A} = \bigcup_{\alpha} \alpha \cdot A_{\alpha}$$

- Which means that, any fuzzy set can be written as the union for all the values of α of the product between α and the α-cuts of the fuzzy set
- This property is very important and it connects the fuzzy and the crisp sets
- It is also very useful for proving different properties of fuzzy sets (some properties are easier to be proved for crisp sets)

Properties of a fuzzy set: α-level sets

- We will illustrate this property on the example with the comfortable house:
  - $\alpha \cdot A_\alpha$ is the fuzzy set in which each element will hace the membership function equal with α.
  - $0.1 \cdot A_{0.1} = \{(1, 0.1), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0.1)\}$
  - $0.2 \cdot A_{0.2} = \{(2, 0.2), (3, 0.2), (4, 0.2), (5, 0.2), (6, 0.2)\}$
  - $\ldots$
  - $0.8 \cdot A_{0.8} = \{(3, 0.8), (4, 0.8)\}$
  - $1.0 \cdot A_{1.0} = \{(4, 1.0)\}$
  - The union of two or more fuzzy sets is defined as the maximum between their membership function, hence
  - $0.1 \cdot A_{0.1} \cup 0.2 \cdot A_{0.2} \cup \ldots \cup 0.8 \cdot A_{0.8} \cup 1.0 \cdot A_{1.0} =$ $= \{(1, 0.1), (2, max(0.1, 0.2)), (3, max(0.1, 0.2, \ldots, 0.8)),$ $(4, max(0.1, \ldots, 0.8, 1)), \ldots (6, max(0.1, 0.2)\} = \tilde{A}$

Properties of fuzzy sets: convexity

6. Convexity of a fuzzy set
   - A fuzzy set $\tilde{A} \subset X$ is convex if and only if $\forall x_1, x_2 \in X$ and $\forall \lambda \in [0,1]$ the following relation takes place: $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \cdot x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$
   - The expression $\lambda \cdot x_1 + (1-\lambda) \cdot x_2$ describes the segment situated between the points having the abscissa x_1 and x_2
   - The expression $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda) \cdot x_2)$ describes the image of this segment through the function $\mu_{\tilde{A}}(x)$
   - **Equivalently, a fuzzy set $\tilde{A}$ is convex iff all its α-level sets are convex**
   - Which means that, if a fuzzy set is not convex, there exist α-level sets of this fuzzy set that are not convex, i.e., there exist segments $x_1^\alpha x_2^\alpha$ which are "interrupted" (are not continues)

Ex: Represent graphically a continuous and convex fuzzy set and a continuous non-convex fuzzy set.

Properties of fuzzy sets: cardinality

7. Cardinality of a fuzzy set
   - Cardinality of a finite fuzzy set $\tilde{A} \subset X$, denoted $|\tilde{A}|$ is defined as:
   $$|\tilde{A}| = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)$$
   - For a continuous fuzzy set $\tilde{A} \subset X$, its cardinality is defined:
   $$|\tilde{A}| = \int_{X} \mu_{\tilde{A}}(x) dx$$
   if the integral exist

7' Relative cardinality of a fuzzy set
   - Is denoted $||\tilde{A}||$
   - Is defined as $||\tilde{A}|| = \frac{|\tilde{A}|}{|X|}$, if it exists, where X is the universe of discourse for the set $\tilde{A}$

How to chose the membership functions

- Like in other aspects of the fuzzy sets theory, there are no clear "recipes" for choosing the membership functions of the fuzzy sets
- If we want to reduce the computations, we will prefer linear membership functions, i.e., triangles and trapeziums
- There are cases when we prefer non-linear membership functions (trigonometric, Gauss-type, etc):
  - There exist researchers that consider that linear membership functions do not provide the best results for some problems, while non-linear functions perform better
  - Sometimes the problem or the domain might need some types of membership functions
  - If we combine fuzzy sets theory with other methods, e.g., neural networks, it can be necessary to use membership functions that are suitable for these methods.

Outline

Operations for fuzzy sets: union, intersection, complement

- Given two fuzzy sets $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$ and $\tilde{B} = \{(x, \mu_{\tilde{B}}(x)) | x \in X\}$ over the same universe of discourse X, we can define operations of union, intersection and complement. We define:
- the *union* of the fuzzy sets $\tilde{A}$ si $\tilde{B}$ as the fuzzy set $\tilde{C} = \tilde{A} \cup \tilde{B}$, given by $\tilde{C} = \{(x, \mu_{\tilde{C}}(x)) | x \in X\}$, where

$$\mu_{\tilde{C}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- the *intersection* of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ as the fuzzy set $\tilde{D} = \tilde{A} \cap \tilde{B}$, given by $\tilde{D} = \{(x, \mu_{\tilde{D}}(x)) | x \in X\}$, where

$$\mu_{\tilde{D}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$$

- the *complement* of $\tilde{A}$ in X as the fuzzy set $\tilde{E} = \mathbb{C}_{\tilde{A}} X$ given by $\tilde{E} = \{(x, \mu_{\tilde{E}}(x)) | x \in X\}$, where

$$\mu_{\tilde{E}}(x) = 1 - \mu_{\tilde{A}}(x)$$

Operations with fuzzy sets: inclusion, equality

- *inclusion* of fuzzy sets: given two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X, the inclusion $\tilde{A} \subseteq \tilde{B}$ takes place iff $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$, $(\forall)x \in X$
- *equality* of two fuzzy sets: two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X are equals iff $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$, $(\forall)x \in X$
- Equivalently, two fuzzy sets $\tilde{A}$ and $\tilde{B}$ included in X are equals iff $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$

Operations with fuzzy sets: examples

1. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "comfortable house for a 4 persons - family" and $\tilde{B} =$ "small house", where
$\tilde{A} = \{(1, 0.1), (2, 0.5), (3, 0.8), (4, 1.0), (5, 0.7), (6, 0.2)\}$ and
$\tilde{B} = \{(1, 1), (2, 0.8), (3, 0.4), (4, 0.1)\}$:
$\tilde{A} \cup \tilde{B} = \{(1, \max(0.1, 1)), (2, \max(0.5, 0.8)), (3, \max(0.8, 0.4)),$
$(4, \max(1, 0.1)), (5, \max(0.7, 0)), (6, \max(0.2, 0)\} =$
$\{(1, 1), (2, 0.8), (3, 0.8), (4, 1), (5, 0.7), (6, 0.2)\}$
$\tilde{A} \cap \tilde{B} = \{(1, \min(0.1, 1)), (2, \min(0.5, 0.8)), (3, \min(0.8, 0.4)),$
$(4, \min(1, 0.1)), (5, \min(0.7, 0)), (6, \min(0.2, 0)\} =$
$\{(1, 0.1), (2, 0.5), (3, 0.4), (4, 0.1), (5, 0), (6, 0)\}$
$\tilde{A} \cup \tilde{B}$ can be read as "comfortable house for a 4 persons - family or small", and $\tilde{A} \cap \tilde{B}$ as "comfortable house for a 4 persons - family and small"

Operations with fuzzy sets: examples (continued)

2. Determine $\mathbb{C}_{\tilde{A}}X$, where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$:
   ("non-comfortable house for a 4 persons - family")
   $\mathbb{C}_{\tilde{A}}X = \{(1, 1 - 0.1), (2, 1 - 0.5), (3, 1 - 0.8), (4, 1 - 1), (5, 1 - 0.7), (6, 1 - 0.2), (7, 1 - 0), (8, 1 - 0), (9, 1 - 0), (10, 1 - 0)\} = \{(1, 0.9), (2, 0.5), (3, 0.2), (4, 0), (5, 0.3), (6, 0.8), (7, 1), (8, 1), (9, 1), (10, 1)\}$
3. Determine the union and intersection of the fuzzy sets $\tilde{A} =$ "real numbers close to 10" and $\tilde{B} =$ "real number considerably larger than 11".
   - Analytically: $\tilde{C} = \tilde{A} \cup \tilde{B}$ si $\tilde{D} = \tilde{A} \cap \tilde{B}$, where
     $\mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$,
     $\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$
     ...
   - Graphically: (more suited in this case), in the next slides:

Example of operations with fuzzy sets: union

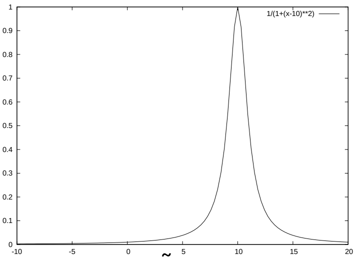

Figure 3: $\tilde{A}$ with $\mu_{\tilde{A}}(x)$

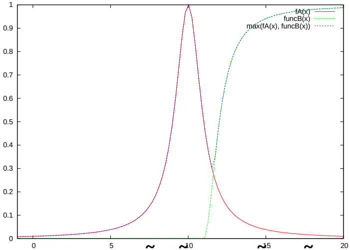

Figure 5: $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$

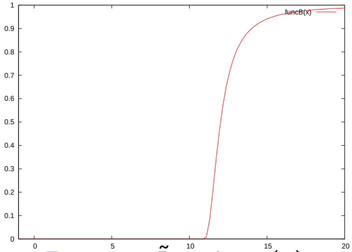

Figure 4: $\tilde{B}$ with $\mu_{\tilde{B}}(x)$

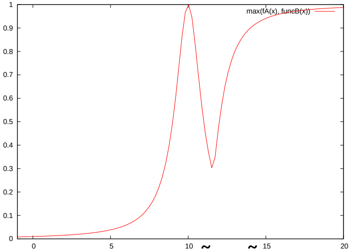

Figure 6: $\tilde{A} \cup \tilde{B}$

Example of operations with fuzzy sets: intersection

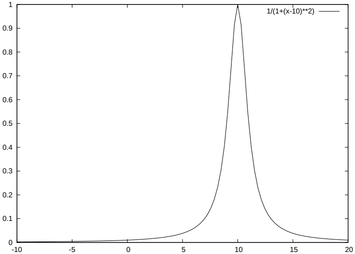

Figure 7: $\mu_{\tilde{A}}(x)$

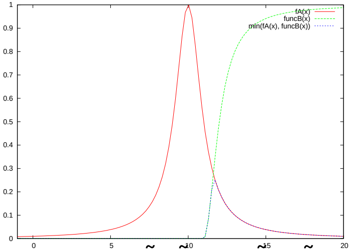

Figure 9: $\tilde{A}$, $\tilde{B}$ and $\tilde{A} \cap \tilde{B}$

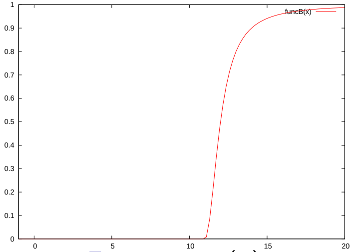

Figure 8: $\mu_{\tilde{B}}(x)$

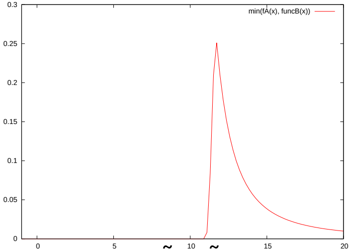

Figure 10: $\tilde{A} \cap \tilde{B}$ (detail)

Example of operations with fuzzy sets: exercises

1. Determine $\mathbb{C}_{\tilde{B}}X$, where $\tilde{B}$ is the fuzzy set "small house", and $X = \{1, 2, \ldots, 9, 10\}$
2. Determine the complement of a fuzzy set that has a continuous trapezoidal-shaped membership function
3. For this fuzzy set, determine the union and intersection between the fuzzy set and its complement. What do you see ?

Outline

Properties of the operations with crisp sets and fuzzy sets

For crisp sets in the universe of discourse X the following properties are true (after [NR74]):

1. Commutativity:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
2. Associativity:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
3. Distributivity:
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
4. Idempotency:
$$A \cup A = A$$
$$A \cap A = A$$

Properties of the operations with crisp sets and fuzzy sets

5. Identity:

$$A \cup \emptyset = \emptyset \cup A = A$$

$$A \cup X = X \cup A = X$$

$$A \cap \emptyset = \emptyset \cap A = \emptyset$$

$$A \cap X = X \cap A = A$$

6. Transitivity: if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
7. Involution: $\overline{\overline{A}} = A$, where $\overline{A} = \mathbb{C}_A X$
8. De Morgan:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Properties of the operations with crisp sets and fuzzy sets

9. Absorption:

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

10. Excluded middle laws *(excluded middle laws)*:

$$A \cup \overline{A} = X$$

$$A \cap \overline{A} = \emptyset$$

- Proprieties 1–9 hold for fuzzy sets, too, but NOT the property 10.
- Some researchers consider this fact (non-fulfillment of the excluded middle laws) as being the main characteristic of fuzzy sets.

Axiomatization of the operations with fuzzy sets

- Bellmann and Giertz proposed a set de axioms (properties) that should be fulfilled by the union, intersection, and complement operations with fuzzy sets.
- They wanted to see if, based on a set of axioms, we can obtain also other operations than maximum for union, minimum for intersection, and $1 - \mu_{\tilde{A}}(x)$ for intersection.
- Bellman and Giertz have shown that only the operators maximum for union, and respectively minimum for intersection fulfill their set of axioms
- However, for complement they could not obtain an unique operator.
- In order to obtain an unique operator for complement, they added the condition that the complement of $1/2$ should be $1/2$.

Conclusions: directions in fuzzy logic

1. The direction followed by mathematicians, who aim to:
   - on the one side, to give a theoretical foundation to the results, operators and formulas from fuzzy logic
   - on the other side, try to extend other domains, mathematical or non-mathematical, through the framework of fuzzy logic.
   - Hence, there exists fuzzy numbers, fuzzy arithmetic, fuzzy functions, fuzzy calculus, fuzzy probabilities, but also fuzzy automata, fuzzy flip-flops, fuzzy codes, fuzzy reliability, etc
2. The second direction is followed by engineers, economists, linguists, medical doctors, etc, who apply the results of fuzzy logic in their domains of activity
   - They must keep themselves informed on the results obtained by mathematicians.

CV Negoița and DA Ralescu.
Mulțimi vagi și aplicațiile lor.
Editura Tehnică, 1974.

H.-J. Zimmermann.
Fuzzy Set Theory – and Its Applications, Second, Revised Edition.
Kluwer Academic Publishers, 1991.