Extensions. The extension principle

Chapter 3

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Outline

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Extensions of fuzzy sets. Type m fuzzy sets

1. Type m fuzzy sets, with $m \geq 2$
   - Definition from previous section corresponds for type 1 fuzzy sets
   - **Definition:** a type 2 fuzzy set is defined as a fuzzy set whose membership values are a type 1 fuzzy set.
   - **Definition:** In a similar manner we can define a type m fuzzy set, with $m \geq 2$, as a fuzzy set whose membership values are a type $m - 1$ fuzzy sets.
   - In practice the fuzzy sets of type greater than 2 or 3 are hardly, if ever, used.
   - The operations with fuzzy sets of type 2 or greater can be defined only using the extension principle.
   - Even for type 2 fuzzy sets, the operations of union, intersection, complement, etc, imply many computations

Extensions of fuzzy sets

2. $\mathbb{L}$-fuzzy sets: fuzzy sets for which the real number interval $[0,1]$ is extended to a set $\mathbb{L}$ named partially ordered set (POS):
   - The interval $[0,1]$ is a POS
3. $\mathbb{B}$-fuzzy sets: similar with $\mathbb{L}$-fuzzy sets, but $\mathbb{B}$ is a o Boole algebra
4. Probabilistic sets (Hirota)
   - Are fuzzy sets with a membership function $\mu_{\tilde{A}}(x, \omega)$
   - For a certain, fixed x, the function becomes a random variable, that has a mean value and a variance
   - The mean value of a probabilistic fuzzy set is a regular fuzzy set
   - The combination between probability and fuzzy can be used, for example, for computing the reliability of some very complex systems (e.g. nuclear plants, in avionics, etc)
   - Fuzzy sets are better suited for modeling the human behaviour (e.g., the apparition of a human error in the operation of a complex system), while probabilities models better the failure of equipments

Extensions of fuzzy sets: IFS – Intuitionistic Fuzzy Sets

5. An intuitionistic fuzzy set (IFS) in univers of discourse X is defined as the triplet:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in X\}$$

$$\mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) : X \to [0,1], 0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$$

- $\mu_{\tilde{A}}(x)$ is named the degree of membership of x to $\tilde{A}$
- $\nu_{\tilde{A}}(x)$ is named the degree of non-membership of x to $\tilde{A}$
- For classic fuzzy sets it holds: $\nu_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$
- Similar with classic fuzzy sets, IFS can be extended to $\mathbb{L} - IFS$ or $\mathbb{B} - IFS$!!
- IFS have been proposed by Atanasov and Stoeva
- For IFS there can be defined operations of union, intersection and complement

Operations with IFS

We present the operations with IFS after [Ata86], [DBR01]

1. $\tilde{A} \subset \tilde{B}$ iff $(\forall)x \in X$ $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x)$
2. $\tilde{A} = \tilde{B}$ iff $\tilde{A} \subset \tilde{B}$ and $\tilde{B} \subset \tilde{A}$.
   Equivalently $\tilde{A} = \tilde{B}$ iff $(\forall)x \in X$ $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x)$
3. $\overline{\tilde{A}} = \{(x, \nu_{\tilde{A}}(x), \mu_{\tilde{A}}(x)) | x \in X\}$
4. $\tilde{A} \cup \tilde{B} = \{(x, \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}) | x \in X\}$
5. $\tilde{A} \cap \tilde{B} = \{(x, \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}) | x \in X\}$
6. $\tilde{A} + \tilde{B} = \{(x, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x), \nu_{\tilde{A}}(x) \cdot \nu_{\tilde{B}}(x)) | x \in X\}$
7. $\tilde{A} \cdot \tilde{B} = \{(x, \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x), \nu_{\tilde{A}}(x) + \nu_{\tilde{B}}(x) - \nu_{\tilde{A}}(x) \cdot \nu_{\tilde{B}}(x)) | x \in X\}$

Outline

Extensions of the operations with fuzzy sets

1. *Cartesian product:* Given the fuzzy sets $\tilde{A}_1, \ldots, \tilde{A}_n$ in the universes $X_1, \ldots, X_n$, the cartesian product of these fuzzy sets is a fuzzy set the product space $X_1 \times \ldots \times X_n$ with the membership function

$$\mu_{\tilde{A}_1 \times \ldots \tilde{A}_n}(x) = min_i[\mu_{\tilde{A}_i}(x_i)]$$

, where $x = (x_1, \ldots, x_n)$, $x_i \in X_i$

   - Example of cartesian product for $n = 2$

2. *The m-th power of a fuzzy set $\tilde{A}$:* in the universe of discourse X is the fuzzy set with the membership function

$$\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m$$

, where $x \in X$

3. Intersection and union operations with fuzzy sets are extended by the triangular, or *t-norms* (for intersection) and *s-norms*, named also *t-co-norms* (the union).

t-norms: definitions

Definition

A *t-norm* is a two valued function $t : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following properties:

1. $t(0,0) = 0$,
   $t(1, \mu_{\tilde{A}}(x)) = t(\mu_{\tilde{A}}(x), 1) = \mu_{\tilde{A}}(x)$, $(\forall)\ x \in X$, i.e.
   $(\forall)\ \mu_{\tilde{A}}(x) \in [0,1]$
2. monotony:
   $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
3. commutativity:
   $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$
4. associativity:
   $t(\mu_{\tilde{A}}(x), t(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$

s-norms: definitions

Definition

An *s-norm (t-conorm)* is a two valued function $s : [0,1] \times [0,1] \to [0,1]$ following properties:

1. $s(1,1) = 1$,
   $s(0, \mu_{\tilde{A}}(x)) = s(\mu_{\tilde{A}}(x), 0) = \mu_{\tilde{A}}(x)$, $(\forall)\ x \in X$, i.e.
   $(\forall)\ \mu_{\tilde{A}}(x) \in [0,1]$
2. monotony:
   $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq s(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$
   if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
3. commutativity:
   $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = s(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$
4. associativity:
   $s(\mu_{\tilde{A}}(x), s(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = s(s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x))$

s-norms and t-norms

- The t-norms and s-norms obeys De Morgan's law, i.e. $A \cap B = \overline{\overline{A} \cup \overline{B}}$:
- $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = 1 - s(1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x))$
- Based on this relation, we can generate a t-norm starting from an s-norm and the opposite, we can generate an s-norm starting from its corresponding t-norm.
- Next, we will give some examples of t-norm, s-norm pairs.

Pairs of t-norms and s-norms

1. Drastic product, drastic sum:

$$t_w(\mu_1(x), \mu_2(x)) = \begin{cases} \mu_1(x) \text{ if } \mu_2(x) = 1 \\ \mu_2(x) \text{ if } \mu_1(x) = 1 \\ 0, \text{ if } 0 \leq \mu_1(x), \mu_2(x) < 1 \end{cases}$$

$$s_w(\mu_1(x), \mu_2(x)) = \begin{cases} \mu_1(x) \text{ if } \mu_2(x) = 0 \\ \mu_2(x) \text{ if } \mu_1(x) = 0 \\ 1, \text{ if } 0 < \mu_1(x), \mu_2(x) \leq 1 \end{cases}$$

Equivalently,

$$t_w(\mu_1, \mu_2) = \begin{cases} \min(\mu_1, \mu_2) \text{ if } \max(\mu_1, \mu_2) = 1 \\ 0, \text{ otherwise} \end{cases}$$

$$s_w(\mu_1, \mu_2) = \begin{cases} \max(\mu_1, \mu_2) \text{ if } \min(\mu_1, \mu_2) = 0 \\ 1, \text{ otherwise} \end{cases}$$

Pairs of t-norms and s-norms

2. Bounded difference, bounded sum:
   $t_1(\mu_1(x), \mu_2(x)) = \max\{0, \mu_1(x) + \mu_2(x) - 1\}$
   $s_1(\mu_1(x), \mu_2(x)) = \min\{1, \mu_1(x) + \mu_2(x)\}$
3. Einstein product, Einstein sum:

$$t_{1.5}(\mu_1(x), \mu_2(x)) = \frac{\mu_1(x) \cdot \mu_2(x)}{2 - [\mu_1(x) + \mu_2(x) - \mu_1(x) \cdot \mu_2(x)]}$$

$$s_{1.5}(\mu_1(x), \mu_2(x)) = \frac{\mu_1(x) + \mu_2(x)}{1 + \mu_1(x) \cdot \mu_2(x)}$$

4. Algebraic product and sum:
   $t_2(\mu_1(x), \mu_2(x)) = \mu_1(x) \cdot \mu_2(x)$
   $s_2(\mu_1(x), \mu_2(x)) = \mu_1(x) + \mu_2(x) - \mu_1(x) \cdot \mu_2(x)$

Pairs of t-norms and s-norms

5. Hamacher product and sum:

$$t_{2.5}(\mu_1(x), \mu_2(x)) = \begin{cases} \frac{\mu_1(x)\cdot\mu_2(x)}{\mu_1(x)+\mu_2(x)-\mu_1(x)\cdot\mu_2(x)}, & \text{if } \mu_1, \mu_2 \neq 0 \\ 0, & \text{if } \mu_1 = \mu_2 = 0 \end{cases}$$

$$s_{2.5}(\mu_1(x), \mu_2(x)) = \begin{cases} \frac{\mu_1(x)+\mu_2(x)-2\cdot\mu_1(x)\cdot\mu_2(x)}{1-\mu_1(x)\cdot\mu_2(x)}, & \text{if } \mu_1, \mu_2 \neq 1 \\ 1, & \text{if } \mu_1 = \mu_2 = 1 \end{cases}$$

6. minimum and maximum:
   $t_3(\mu_1(x), \mu_2(x)) = \min\{\mu_1(x), \mu_2(x)\}$
   $s_3(\mu_1(x), \mu_2(x)) = \max\{\mu_1(x), \mu_2(x)\}$

t-norms and s-norms

The operators presented before satisfy the following relations:

$$t_w \leq t_1 \leq t_{1.5} \leq t_2 \leq t_{2.5} \leq t_3$$

$$s_3 \leq s_{2.5} \leq s_2 \leq s_{1.5} \leq s_1 \leq s_w$$

More general, Dubois and Prade have shown that each t-norms is bounded between the drastic product and by minimum, and that any s-norm is situated between the maximum and the drastic sum.

Which means, for all t-norm t and for all s-norm s are satisfied the following relations:

$$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \;\; x \in X$$

$$\max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \leq s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq s_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \; x \in X$$

Parametrized intersection and union operators: proposed by Hamacher

- Hamacher proposed the following operators for intersection and union:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)}{\gamma + (1-\gamma)\cdot(\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x))},\ \gamma \geq 0$$

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \frac{(\gamma'-1)\cdot\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)+\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)}{1+\gamma'\cdot\mu_{\tilde{A}}(x)\cdot\mu_{\tilde{B}}(x)},\ \gamma' \geq -1$$

- For $\gamma = 0$ and $\gamma' = -1$ we obtain Hamacher product and sum, and for $\gamma = 1$ and $\gamma' = 0$ we obtain algebraic product and sum.

Parametrized intersection and union operators: proposed by Yager

- Yager proposed the following operators for intersection and union:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = 1 - \min\{1, [(1-\mu_{\tilde{A}}(x))^p + (1-\mu_{\tilde{B}}(x))^p]^{1/p}\},\ p \geq 1$$

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \min\{1, [(\mu_{\tilde{A}}(x))^p + (\mu_{\tilde{B}}(x))^p]^{1/p}\},\ p \geq 1$$

- For $p = 1$ we obtain bounded difference and sum, while for $p \to \infty$, Yager's operators converge to *minimum* and respectively *maximum*
- Yager's operators satisfy De Morgan laws, are commutative and associative for all p, monotone non-decreasing in $\mu(x)$ and include the classic operators from dual (clasic) logic
- However, Yager's operators are not distributive

Averaging operators

- In *decision making* or *multi-criteria decision theory* we want many times to realize trade-offs between *conflicting goals*
- The solution will be situated between the most optimistic *(ower bound* and the most pessimistic *upper bound* values.
- These operators are named *averaging operators*
- Examples of averaging operators: "fuzzy and" and "fuzzy or", defined as:
  $\mu_{\widetilde{and}}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) =$
  $\gamma \cdot \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \frac{(1-\gamma)\cdot(\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x))}{2}$, $x \in X$, $\gamma \in [0, 1]$
  $\mu_{\widetilde{or}}(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) =$
  $\gamma \cdot \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \frac{(1-\gamma)\cdot(\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x))}{2}$, $x \in X$, $\gamma \in [0, 1]$

Averaging operators

- The operators “fuzzy and” and “fuzzy or” combine minimum, respectively maximum, with arithmetic mean
- In this way it is realised the compensation of the values of the membership functions for aggregate sets.
- Using empirical data, Zimmermann claims that these operators give good results for certain applications ([Zim91], p 36)
- There exists other averging operators !
- Zimmermann proposed also a parametrized averaging operator, called “compensatory and”, for m fuzzy sets:

$$\mu_{\tilde{A}_i,comp} = (\prod_{i=1}^{m} \mu_i(x))^{(1-\gamma)} \cdot (1 - \prod_{i=1}^{m}(1 - \mu_i(x)))^{\gamma}$$

, $x \in X, 0 \leq \gamma \leq 1$

Averaging operators

- The aggregation of two fuzzy sets, $\tilde{A}$ and $\tilde{B}$, can be realized by a linear combination of minimum and maximum:
  $\mu_1(\mu_{\tilde{A}(x)}, \mu_{\tilde{B}}(x)) =$
  $\gamma \cdot \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + (1-\gamma) \cdot \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},$
  $\gamma \in [0,1]$
- min and max can be replaced with algebraic product and algebraic sum
- Zimmermann and Zysno claim that the operator "compensatory and" is better suited than other operators for "human decision making" problems, [Zim91], p38.
- Being so many operators, the question that arises is: *what operators to use ?*
- We will present criteria for selecting appropriate aggregation operators

Outline

Criteria for selecting the operators

1. *Axiomatic strength:* if all other characteristics are equal, is better the operator which satisfies the less restrictive (limiting) set of axioms
2. *Empirical fit:* it is important that the chosen operators to be appropriate for the domain where they are used. This can be verified only through empiric tests .
3. *Adaptability:* if one wants to have a small number of operators to model many situations, then it is recommended to use parametrized operators (Yager, Hamacher). Of course, these operators are not numerically efficient (they require complex computations)
4. *Computational (numerical) efficiency:* it is obvious that *max* and *min* operators imply less computational effort than parametrized operators, hence, they will be preferred if we want a good numerical efficiency.

Criteria for selecting the operators

5. *Compensation:* Compensation is defined:
   given $k \in [0,1]$, if $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = k$, operator t is compensatory if, modifying $\mu_{\tilde{A}}(x)$, it can be obtained $t(\mu_{\tilde{A}}(x_k), \mu_{\tilde{B}}(x_k)) = k$ for another $\mu_{\tilde{B}}(x)$.

Example:
If $\mu_{\tilde{A}} = 0.2$ and $\mu_{\tilde{B}} = 0.3$, if the operator t is min, we obtain $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min(0.2, 0.3) = 0.2$ (hence $k = 0.2$)
If we make $\mu_{\tilde{A}} = 0.1$, then, no matter what would be the value of $\mu_{\tilde{B}}(x)$, we **cannot** obtain $\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = 0.2$ because $\min(0.1, \mu_{\tilde{B}}(x)) \leq 0.1$, hence the operator min is not compensatory.

If the operator t is the algebraic product, then we obtain: $0.2 \cdot 0.3 = 0.06$, hence $k = 0.06$ and if we make $\mu_{\tilde{A}}(x) = 0.1$, we have $0.1 \cdot \mu_{\tilde{B}}(x) = 0.06 \implies \mu_{\tilde{B}}(x) = \frac{0.06}{0.1} = 0.6$, hence the operator algebraic product is compensatory.

Criteria for selecting the operators

- In general, if we want to solve a problem where the time constrains are important, then the numerical efficiency criterion is of prime importance, because we want to obtain the results in real time. This is the case, e.g., in control engineering
- If, on the other side, we solve a complex problem, e.g., an expert system for medical diagnosis, then the complexity and the "finesse" of the operator is more important than to obtain quickly the result.
- Beside the above criteria, we can consider other criteria as well, e.g., *technological fit*: if the fuzzy operations are implemented in hardware, then some operators can be easier implemented in certain technologies.

Outline

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The importance of the extension principle

- In its general form has been formulated by Zadeh in 1973
- It is very important because it can be used to extend different mathematical and non-mathematical theories or domains through the framework of fuzzy logic (to be "fuzzified")
- Hence, there exist fuzzy numbers, fuzzy arithmetic, fuzzy analysis, fuzzy probabilities. but also fuzzy codes, fuzzy automata, fuzzy flip-flops, etc

Functions

- We recall that a function f is an application $f : X \to Y$ that associates to each $x \in X$ an unique $y \in Y$.
- That is, $(\forall) x \in X$ $(\exists)$ an unique $y \in Y$ such that $y = f(x)$, named the image of x through the function f.
- This means that:
  1. No $x \in X$ has two or more images, e.g. y_1 si y_2
  2. No x from X has 0 (zero) images through function f.
- FIGURE !!!!

Function

- A function is *injective*, if $(\forall)\ x_1, x_2 \in X$ with $x_1 \neq x_2$, it holds $f(x_1) \neq f(x_2)$ (every y from Y is the image of at most one x from X)
- A function is *surjective*, if $(\forall) y \in Y$ $(\exists) x \in X$ such that $y = f(x)$ (every y from Y is the image of at least one x from X)
- A function is *bijective* iff it is both injective and surjective (every y from Y is the image of *exactly one* x from X)
- In other words, to every x in X it corresponds an *unique* y in Y, or, between the sets X and Y it can be established a "one to one" correspondence.
- A function is invertible iff it is bijective.
- The Inverse of a function f is denoted cu f^{-1}: $f^{-1} : Y \rightarrow X$ such that $x = f^{-1}(y)$
- In the following, we will make an abuse of notation, i.e., we will denote $x = f^{-1}(y)$ even if the function f is not invertible.

Outline

The extension principle in its reduced form

Definition: Let X, Y, be universes, $\tilde{A} \subset X$ be a fuzzy set in X and let $f : X \to Y$ be a function (a mapping) such that $y = f(x)$.
The extension principle allows the definition of a fuzzy set $\tilde{B} \subset Y$, $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x),\ x \in X\}$, where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x), & \text{if } (\exists)\ f^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

Explanations

If we have a fuzzy set $\tilde{A} \subset X$ and a function $f : X \to Y$, the extension principle allow us to determine the fuzzy set $\tilde{B} \subset Y$ which is the image (the mapping) of the set $\tilde{A}$ through the function f. The following situations may appear:

- If an element $y \in Y$ is the image of a *unique* element $x \in X$, then it is straight forward to consider $\mu_{\tilde{B}}(y) = \mu_{\tilde{A}}(x)$
- If an element $y \in Y$ is the image of *no one* element $x \in X$, then it would be normal to consider that $\mu_{\tilde{B}}(y) = 0$
- If an element $y \in Y$ is the image of several elements $x \in X$ (for example $x_i, x_j, \ldots, x_k$), then, the degree of membership of y la $\tilde{B}$ will be the maximum between the degrees of membership of the elements $x_i, x_j, \ldots, x_k$ to $\tilde{A}$

In formula, by $x \in f^{-1}(y)$ are denoted those elements x form X whose image through function f is y from Y, i.e. $y = f(x)$.

The extension principle in its general form

Definition: Let $X = X_1 \times X_2 \times \ldots \times X_n$ be the cartesian product of the universes X_i, $i = 1, \ldots, n$, and $\tilde{A}_i \subset X_i$ be fuzzy sets in X_i, and let $f : X \to Y$ be a function (a mapping) such that $y = f(x_1, x_2, \ldots, x_n)$, $x_i \in X_i$, $i = 1, \ldots, n$, and Y is also an universe.

The extension principle permits the definition of a fuzzy set $\tilde{B} \subset Y$, $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y = f(x_1, x_2, \ldots, x_n),\ (x_1, x_2, \ldots, x_n) \in X\}$, where:

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} min(\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)), & \text{if } (\exists)\ f^{-1}(y) \\ 0, & \text{otherwise} \end{cases}$$

where $x = (x_1, x_2, \ldots, x_n)$

Explanations

- In the general form, the universe X is the cartesian product of the universes $X_1, X_2, \ldots X_n$, and the fuzzy set $\tilde{A}$ is the cartesian product of the fuzzy sets $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$, i.e. $\tilde{A} = \tilde{A}_1 \times \tilde{A}_2 \times \ldots \times \tilde{A}_n$
- According to the formula for the cartesian product, it results that $\mu_{\tilde{A}}(x) = \min(\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n))$, where $x = (x_1, x_2, \ldots, x_n)$
- Of course, the extension principle in the reduced form can be obtained from the extension principle in the general form, by making $n = 1$.

Outline

Applications of the extension principle in the reduced form

Let $X = Y = \mathbb{Z}$, the set of integer numbers, let $\tilde{A} \subset \mathbb{Z}$, $\tilde{A} = \{(-2, 0.3), (-1, 0.5), (0, 0.8), (1, 1), (2, 0.7), (3, 0.1)\}$ and let the function $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$. Which is the fuzzy set $\tilde{B} \subset \mathbb{Z}$ with $\tilde{B} = f(\tilde{A})$ (i.e. $\tilde{B}$ is the image of the set $\tilde{A}$ through the function f) ?.

- $\mu_{\tilde{B}}(0) = \mu_{\tilde{A}}(0) = 0.8$ and $\mu_{\tilde{B}}(9) = \mu_{\tilde{A}}(3) = 0.1$
- $\mu_{\tilde{B}}(1) = \sup_{x \in f^{-1}(1)}(\mu_{\tilde{A}}(x)) = \sup(\mu_{\tilde{A}}(-1), \mu_{\tilde{A}}(1))$ $= \sup(0.5, 1) = 1$
- $\mu_{\tilde{B}}(4) = \sup_{x \in f^{-1}(4)}(\mu_{\tilde{A}}(x)) = \sup(\mu_{\tilde{A}}(-2), \mu_{\tilde{A}}(2))$ $= \sup(0.3, 0.7) = 0.7$
- If we consider that all the integer numbers between 0 and 9belong to the set $\tilde{B}$, then: $\mu_{\tilde{B}}(2) = \mu_{\tilde{B}}(3)$ $= \mu_{\tilde{B}}(5) = \mu_{\tilde{B}}(6) = \mu_{\tilde{B}}(7) = \mu_{\tilde{B}}(8) = 0$ because $2, 3, 5, 6, 7$ si 8 are not the image of any element from $\tilde{A}$ through the function f.
- In conclusion, $\tilde{B} = \{(0, 0.8)(1, 1), (2, 0), (3, 0), (4, 0.7), (5, 0), (6, 0), (7, 0), (8, 0), (9, 0.1)\}$

Applications of the extension principle: the addition of two discrete fuzzy numbers

Given $X_1 = X_2 = Y = \mathbb{Z}$ and the fuzzy sets $\tilde{A}_1 \subset X_1$ *approximately 2* and $\tilde{A}_2 \subset X_2$ *approximately 6* described as:
$\tilde{A}_1 = \{(1, 0.2), (2, 1), (3, 0.5), (4, 0.1)\}$,
$\tilde{A}_2 = \{(5, 0.2), (6, 1), (7, 0.5), (8, 0.1)\}$.
We want to obtain the fuzzy set $\tilde{B} \subset Y$ given by $\tilde{B} = \tilde{A}_1 \oplus \tilde{A}_2$, where the symbol $\oplus$ represents the addition of the fuzzy numbers, defined as follows:

$$\mu_{\tilde{B}}(y) = \sup_{y = x_1 + x_2} (\min(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2)))$$

Answer:
$\tilde{B} = \{(6, 0.2), (7, 0.2), (8, 1), (9, 0.5), (10, 0.5), (11, 0.1), (12, 0.1)\}$

- We notice that we obtain the number *approximately 8*, as expected, but the "width" of the sum (i.e., the degree of imprecision) is bigger than the degree of imprecision of the terms (the added fuzzy numbers).

Addition of two discrete fuzzy numbers

Example: computation of $\mu_{\tilde{B}}(9)$:

- We start from $y = x_1 + x_2$ which, for $y = 9$, becomes:
  - $9 = 1 + 8$
  - $9 = 2 + 7$
  - $9 = 3 + 6$
  - $9 = 4 + 5$
- Replacing in formula $\mu_{\tilde{B}}(y) = \sup_{y=x_1+x_2}\{\min(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2))\}$ we obtaine:
- $\mu_{\tilde{B}}(9) = \sup_{9=x_1+x_2}\{\min(\mu_{\tilde{A}_1}(1), \mu_{\tilde{A}_2}(8))), \min(\mu_{\tilde{A}_1}(2), \mu_{\tilde{A}_2}(7))), \min(\mu_{\tilde{A}_1}(3), \mu_{\tilde{A}_2}(6))), \min(\mu_{\tilde{A}_1}(4), \mu_{\tilde{A}_2}(5)))\} = \sup\{\min(0.2, 0.1), \min(1, 0.5), \min(0.5, 1), \min(0.1, 0.2)\} = \sup(0.1, 0.5, 0.5, 0.1) = 0.5$

Outline

Operations for type 2 fuzzy sets

- The union, intersection and complement operations for type 2 fuzzy sets can be defined *only* by using the extension principle!
- We will consider only type 2 fuzzy sets with discrete domains
- Let two fuzzy sets of type 2 defined by: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$ and $\tilde{B} = \{(x, \mu_{\tilde{B}}(x))\}$, where:

$$\mu_{\tilde{A}}(x) = \{(u_i, \mu_{u_i}(x)) \mid x \in X, u_i, \mu_{u_i}(x) \in [0, 1]\}$$

$$\mu_{\tilde{B}}(x) = \{(v_j, \mu_{v_j}(x)) \mid x \in X, v_j, \mu_{v_j}(x) \in [0, 1]\}$$

Operations for type 2 fuzzy sets: Union

Definition

For the fuzzy sets of type 2 defined above, the membership function of their *union* is defined as:

$\mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cup \mu_{\tilde{B}}(x) =$

$$= \{(w, \mu_{\tilde{A}\cup\tilde{B}}(w)) \mid w = \max\{u_i, v_j\}, u_i, v_j \in [0,1]\}$$

where

$$\mu_{\tilde{A}\cup\tilde{B}}(w) = \sup_{w=\max\{u_i, v_j\}} \min\{\mu_{u_i}(x), \mu_{v_j}(x)\}$$

Operations for type 2 fuzzy sets: Intersection and Complement

Definition

For the fuzzy sets of type 2 defined above, the membership function of their *intersection* is defined as:

$\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x) =$

$$= \{(w, \mu_{\tilde{A}\cap\tilde{B}}(w)) \,|\, w = \min\{u_i, v_j\}, u_i, v_j \in [0,1]\}$$

where

$$\mu_{\tilde{A}\cap\tilde{B}}(w) = \sup_{w=\min\{u_i,v_j\}} \min\{\mu_{u_i}(x), \mu_{v_j}(x)\}$$

Definition

The complement of $\tilde{A}$ is defined by: $\mu_{\mathbb{C}\tilde{A}}(x) = \{[(1-u_i), \mu_{\tilde{A}}(u_i)]\}$

Operations for type 2 fuzzy sets: example

- Let $X = 1, 2, \ldots, 10$ the universe of discourse
- and the fuzzy sets $\tilde{A} =$ *small integers* and $\tilde{B} =$ *integers close to* 4 defined by:
- $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$, $\tilde{B} = \{(x, \mu_{\tilde{B}}(x))\}$
- where, for $x = 3$ we have:
  $\mu_{\tilde{A}}(3) = \{(u_i, \mu_{u_i}(3)) \,|\, i = 1, \ldots, 3\} = \{(.8, 1), (.7, .5), (.6, .4)\}$
  $\mu_{\tilde{B}}(3) = \{(v_j, \mu_{v_j}(3)) \,|\, j = 1, \ldots, 3\} = \{(1, 1), (.8, .5), (.7, .3)\}$
- Compute $\mu_{\tilde{A} \cap \tilde{B}}(3)$

Operations for type 2 fuzzy sets: example

u_i	v_j	$w = \min\{u_i, v_j\}$	$\mu_{u_i}(3)$	$\mu_{v_j}(3)$	$\min\{\mu_{u_i}(3), \mu_{v_j}(3)\}$
.8	1	.8	1	1	1
.8	.8	.8	1	.5	.5
.8	.7	.7	1	.3	.3
.7	1	.7	.5	1	.5
.7	.8	.7	.5	.5	.5
.7	.7	.7	.5	.3	.3
.6	1	.6	.4	1	.4
.6	.8	.6	.4	.5	.4
.6	.7	.6	.4	.3	.3

Table 1: Example of intersection of type 2 fuzzy sets

Operations for type 2 fuzzy sets: example

Next, we have to compute the supremum of the degrees of membership of all pairs (u_i, v_j) which yield w as minimum:

$$\sup_{.8=\min\{u_i,v_j\}} \{1, .5\} = 1$$

$$\sup_{.7=\min\{u_i,v_j\}} \{.3, .5, .5, .3\} = .5$$

$$\sup_{.6=\min\{u_i,v_j\}} \{.4, .4, .3\} = .4$$

Hence, the membership function of the intersection of fuzzy sets $\tilde{A}$ and $\tilde{B}$, for $x = 3$ is:

$$\mu_{\tilde{A}\cap\tilde{B}}(3) = \{(.8, 1), (.7, .5), (.6, .4)\}$$

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