Measure of fuzziness

or

How “fuzzy” is a fuzzy set ?

Doru Todinca

Departament of Computers and Information Technology
UPT

Measure of fuzziness

- Researchers aimed to find a *measure of fuzziness* (or *MOF*), which is a measure of the degree of fuzziness of a fuzzy set.
- Hence, *MOF* indicates the degree of fuzziness of a fuzzy set
- The problem has been approached in several manners:
  1. Using the concep of entropy, introduced by Claude Shannon (de Luca and Termini)
  2. Starting from the fact that, in general, the fuzzy sets do not comply to the excluded middle laws. More exactly, the MOF was based on the distinction between a set and its complement (the smaller the difference the "more fuzzy" is the fuzzy set !)
- Next, we will discuss the two approaches, for the case when the support of the fuzzy set $\tilde{A}$ is finite.

Entropy-based MOF

Let $\mu_{\tilde{A}}(x)$ be the memebership function of the fuzzy set $\tilde{A}$ for $x \in X$, X finite. A function $d(\tilde{A})$ measure of fuzziness will have the following properties:

1. $d(\tilde{A}) = 0$ if $\tilde{A}$ is a crisp set in X
2. $d(\tilde{A})$ has a unique maximum for the fuzzy set with $\mu_{\tilde{A}}(x) = \frac{1}{2}\ (\forall)x \in X$
3. $d(\tilde{A}) \geq d(\tilde{A}')$ daca $\tilde{A}'$ is "more crisp" than $\tilde{A}$, i.e, either $\mu_{\tilde{A}'}(x) \leq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) \leq \frac{1}{2}$, or $\mu_{\tilde{A}'}(x) \geq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) \geq \frac{1}{2}$
4. $d(\mathbb{C}\tilde{A}) = d(\tilde{A})$

MOF 1, based on entropy

Definition

The entropy, as a measure of fuzziness is defined as:

$$d(\tilde{A}) = H(\tilde{A}) + H(\mathbb{C}\tilde{A})$$

, $x \in X$, where:

$$H(\tilde{A}) = -K \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) \cdot \ln(\mu_{\tilde{A}}(x_i))$$

, where n is the number of elements in the support of the fuzzy set $\tilde{A}$ and K is a positive constant.

MOF 1, based on entropy

Using Shannon's function $S(x) = -x \cdot \ln(x) - (1-x) \cdot \ln(1-x)$, the previous expression can be simplifies as follows:

Definition

Entropy d as a measure of fuzziness of a fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)\}$ is defined by:

$$d(\tilde{A}) = K \cdot \sum_{i=1}^{n} S(\mu_{\tilde{A}}(x_i))$$

MOF 2

In order to measure the distance between a fuzzy set and its complement, Yager suggested the following metric:

Definition

$$D_p(\tilde{A}, \mathbb{C}\tilde{A}) = \left[\sum_{i=1}^{n} |\mu_{\tilde{A}}(x_i) - \mu_{\mathbb{C}\tilde{A}}(x_i)|^p \right]^{1/p}$$

, with $p = 1, 2, 3 \ldots$

MOF 2

If we denote $S = supp(\tilde{A})$ the suport of the fuzzy set $\tilde{A}$, then $D_p(S, \mathbb{C}S) = |S|^{1/p}$

Definition

A MOF for the set $\tilde{A}$ can be defined as:

$$f_p(\tilde{A}) = 1 - \frac{D_p(\tilde{A}, \mathbb{C}\tilde{A})}{|supp(\tilde{A})|^{1/p}}$$

$f_p(\tilde{A}) \in [0, 1]$ and $f_p(\tilde{A})$ satisfies also the proprerties 1–4 required by de Luca and Termini.

MOF 2

For $p = 1$ $D_p(\tilde{A}, \mathbb{C}\tilde{A}))$ yields the Hamming metric (distance):

$$D_1(\tilde{A}, \mathbb{C}\tilde{A}) = \sum_{i=1}^{n} |\mu(x_i) - \mu_{\mathbb{C}\tilde{A}}(x_i)|$$

Since $\mu_{\mathbb{C}\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$, it results that

$$D_1(\tilde{A}, \mathbb{C}\tilde{A}) = \sum_{i=1}^{n} |2 \cdot \mu_{\tilde{A}}(x_i) - 1|$$

MOF 2

For $p = 2$ we obtain an Euclidean metric :

$$D_2(\tilde{A}, \mathbb{C}\tilde{A}) = \left(\sum_{i=1}^{n} (\mu_{\tilde{A}}(x_i) - \mu_{\mathbb{C}\tilde{A}}(x_i))^2 \right)^{1/2}$$

Since $\mu_{\mathbb{C}\tilde{A}} = 1 - \mu_{\tilde{A}}$, we have:

$$D_2(\tilde{A}, \mathbb{C}\tilde{A}) = \left(\sum_{i=1}^{n} (2 \cdot \mu_{\tilde{A}}(x_i) - 1)^2 \right)^{1/2}$$