# Elements of fuzzy arithmetic

## Chapter 7

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# Outline

# Sources

This lecture contains text, figures, formulae, etc, taken (and adapted) from the book Arnold Kaufmann, Madan M. Gupta, “Introduction to fuzzy arithmetic: theory and applications”, Van Nostrand Reinhold, 1991, [KG91].

# Outline

Confidence intervals

Fuzzy numbers

Operations with fuzzy numbers
- Addition of fuzzy numbers
- Subtraction of fuzzy numbers
- Multiplication of fuzzy numbers
- Division of fuzzy numbers
- Examples of operations with fuzzy numbers

# Confidence intervals: definitions

- In many practical situations we can say about a value only that it is situated inside a closed interval in $\mathbb{R}$, i.e., between two values, $a_1, a_2 \in \mathbb{R}$, $a_1 \leq a_2$
- *Definition:* such an interval of real numbers is called *confidence interval* or *interval of confidence*, and is denoted $A = [a_1, a_2]$
- A confidence interval can be open at left (e.g., $A = (a_1, a_2]$), open at right, or open at left and right (i.e., open, e.g., $A = (a_1, a_2)$)
- or it is possible that $a_1 = -\infty$ and/or $a_2 = +\infty$

# Operations with confidence intervals: addition and subtraction

- If we know that $x \in [a_1, a_2]$, $y \in [b_1, b_2]$ (where $A = [a_1, a_2]$ and $B = [b_1, b_2]$), what can we say about $x + y,\ x - y,\ x \cdot y, \frac{x}{y}$ ? (about $x \cdot y$ and $\frac{x}{y}$ we can discuss only in $\mathbb{R}^+$)
- Evidently, $x + y \in [a_1 + b_1, a_2 + b_2], x - y \in [a_1 - b_2, a_2 - b_1]$
- Notations: $A(+)B = [a_1 + b_1, a_2 + b_2]$, $A(-)B = [a_1 - b_2, a_2 - b_1]$
- A particular case of subtraction is the image (the opposite) of a confidence interval $A$, given by $A^- = [-a_2, -a_1]$
- We note that $A(+)A^- = [a_1 - a_2, a_2 - a_1] \neq 0$ in general, where the interval 0 is defined $0 = [0, 0]$
- In general any real number $t \in \mathbb{R}$ can be represented as a confidence interval in the form $[t, t]$
- The set of the interval of confidence in $\mathbb{R}$ is associative, commutative and has a neutral element ($0 = [0, 0]$) for addition, but the image is not symmetrical.

# Operations with confidence intervals: multiplication and division

- The multiplication and division of intervals of confidence in $\mathbb{R}^+$ are defined as:
  $A(\cdot)B = [a_1, a_2] \cdot [b_1, b_2] = [a_1 \cdot b_1, a_2 \cdot b_2]$
  $A(:)B = [a_1, a_2](:)[b_1, b_2] = [\frac{a_1}{b_2}, \frac{a_2}{b_1}]$, where $b_1, b_2 \neq 0$
- The inverse of the interval $A = [a_1, a_2]$ is defined as $A^{-1} = [\frac{1}{a_2}, \frac{1}{a_1}]$, where $a_1, a_2 \neq 0$
- The set of confidence intervals in $\mathbb{R}^+$ is associative, commutative, has a neutral element ($1 = [1, 1]$) for multiplication, but it is not symmetrical because $A(:)A = A(\cdot)A^{-1} = [a_1, a_2](:)[a_1, a_2] = [\frac{a_1}{a_2}, \frac{a_2}{a_1}] \neq [1, 1] = 1$

# Outline

# Fuzzy numbers: definitions

## Definition

A *fuzzy number*, or *uncertain number*, in $\mathbb{R}$ is a fuzzy subset in $\mathbb{R}$ which is normal and convex.

- Kaufmann and Gupta consider a fuzzy number as the association between confidence intervals and *presumption levels, (levels of presumptions)*:
- It is considered that for $\alpha = 1$ we have the maximum presumption about the value of the fuzzy number, and for $\alpha = 0$ we have the minimum level of presumption.
- $\forall \alpha \in [0, 1]$ it is possible to establish a level of presumption $A^{\alpha} = [a_1^{(\alpha)}, a_2^{(\alpha)}]$
- It must be fulfilled the condition that if $\alpha$ increases, the interval of confidence never increases, i.e., if $\alpha_1 \leq \alpha_2$ then $[a_1^{(\alpha_2)}, a_2^{(\alpha_2)}] \subseteq [a_1^{(\alpha_1)}, a_2^{(\alpha_1)}]$

# Outline

# General considerations

- The operations with fuzzy numbers will be defined with two methods, which are equivalent:
  - Using the extension principles
  - Using the presumption levels, which are equivalent with the $\alpha$-cuts ($\alpha$-level sets) of a fuzzy number.
- It can be proved that the two approaches (methods) for the definition of the operations with fuzzy numbers are equivalent.
- In the proof it is used the fact that any fuzzy set is the union of all its $\alpha$-level sets, with $\alpha \in [0, 1]$

# Outline

# Addition of fuzzy numbers

## Definition

Given two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ in $\mathbb{R}$, their sum $\tilde{A} \oplus \tilde{B}$ is defined as: $\forall x, y, z \in \mathbb{R}$

$$\mu_{\tilde{A}\oplus\tilde{B}}(z) = \sup_{z=x+y} (min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)))$$

, or, using the notations from [KG91],
$\mu_{\tilde{A}\oplus\tilde{B}}(z) = \bigvee_{z=x+y} (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y))$

## Definition

Using $\alpha$-cuts, the addition of two fuzzy numbers is defined:
$A^\alpha(+)B^\alpha = [a_1^\alpha, a_2^\alpha](+)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha]$

The operations from the first definition are used for discrete fuzzy numbers (for example in $\mathbb{Z}$), while the second definition is used for continuous fuzzy numbers (in $\mathbb{R}$).

# Outline

# Subtraction of fuzzy numbers

Is defined:
$\forall x, y, z \in \mathbb{R}$

$$\mu_{\tilde{A} \ominus \tilde{B}}(z) = \sup_{z=x-y} (min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)))$$

or:
$A^\alpha(-)B^\alpha = [a_1^\alpha, a_2^\alpha](-)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha]$ because
$B^- = [b_1^\alpha, b_2^\alpha]^- = [-b_2^\alpha, -b_1^\alpha]$

# Outline

# Multiplication of fuzzy numbers

Is defined in $\mathbb{R}^+$:
$\forall x, y, z \in \mathbb{R}^+$

$$\mu_{\tilde{A} \odot \tilde{B}}(z) = \sup_{z = x \cdot y} \left( min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)) \right)$$

or:
$A^\alpha(\cdot)B^\alpha = [a_1^\alpha, a_2^\alpha](\cdot)[b_1^\alpha, b_2^\alpha] = [a_1^\alpha \cdot b_1^\alpha, a_2^\alpha \cdot b_2^\alpha]$

# Outline

# Division of fuzzy numbers

Is defined in $\mathbb{R}^+$:
$\forall x, y, z \in \mathbb{R}^+$

$$\mu_{\tilde{A} \oslash \tilde{B}}(z) = \sup_{z=x/y} (min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)))$$

or:
$A^\alpha(/)B^\alpha = [a_1^\alpha, a_2^\alpha](/)[b_1^\alpha, b_2^\alpha] = [\frac{a_1^\alpha}{b_2^\alpha}, \frac{a_2^\alpha}{b_1^\alpha}]$ because
$B_\alpha^{-1} = [b_1^\alpha, b_2^\alpha]^{-1} = [\frac{1}{b_2^\alpha}, \frac{1}{b_1^\alpha}]$

# Special fuzzy numbers and algebraic properties

- A singleton $t$ can be extended to a fuzzy number $t$
- Special fuzzy numbers are 0 si 1
- The set of fuzzy numbers in $\mathbb{R}$ is associative and commutative for addition, there is a neutral (the number 0), but the image is not symmetric because in general $\tilde{A} \ominus \tilde{A} \neq 0$
- The multiplication of fuzzy numbers in $\mathbb{R}^+$ is associative, commutative, there is a neutral (the number 1), but the inverse is not symmetric, because in general $\tilde{A} \oslash \tilde{A} \neq 1$

# Outline

# Operations with discrete fuzzy numbers

- For operations with discrete fuzzy numbers it is applied the first set of formulae (the first method) and we work based on *extension principle* (like in the exemple from Chapter 3, from the Extension principle).
- For addition and subtraction we do like in that example: we look for all pairs of numbers $x$ and $y$ whose sum or difference is equal with $z$, and we apply the extension principle
- For multiplication (only for positive numbers), at the left of the normal value (for which $\mu = 1$) we take into account all couplets where $x \cdot y \leq z$, and at the right of the normal value we take into account all couplets for which $x \cdot y \geq z$. Also, we compute values of z for which $\mu = 1$.
- It will result a number which is increasing (more precisely, non-decreasing) to the left of the normal value, and decreasing (non-increasing) to the right of the normal value.

# Operations with continuous fuzzy numbers in $\mathbb{R}$, respectively in $\mathbb{R}^+$

- For continuous fuzzy numbers we work with $\alpha$-level sets
- In general a fuzzy number is given as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x \leq l_1 \\ f_1(x), & \text{if } l_1 \leq x \leq l_2 \\ f_2(x), & \text{if } l_2 \leq x \leq l_3 \\ 0, & \text{if } l_3 \leq x \end{cases}$$

  , where $f_1$ is an increasing function, and $f_2$ is a decreasing function.
- Figure !
- We make $\alpha = f_1(a_1^\alpha)$ and it results $a_1^\alpha = f_1^{-1}(\alpha)$ , and similar from $\alpha = f_2(a_2^\alpha)$ it results $a_2^\alpha = f_2^{-1}(\alpha)$

# Operations with continuous fuzzy numbers in $\mathbb{R}$, respectively in $\mathbb{R}^+$

- The second fuzzy number is given as:

$$\mu_{\tilde{B}}(x) = \begin{cases} 0, & \text{if } x \leq m_1 \\ g_1(x), & \text{if } m_1 \leq x \leq m_2 \\ g_2(x), & \text{if } m_2 \leq x \leq m_3 \\ 0, & \text{if } m_3 \leq x \end{cases}$$

  , where $g_1$ is an increasing, and $g_2$ a decreasing function
- We make $\alpha = g_1(b_1^\alpha)$ and it results $b_1^\alpha = g_1^{-1}(\alpha)$ and similar from $\alpha = g_2(a_2^\alpha)$ it results $b_2^\alpha = g_2^{-1}(\alpha)$
- Then we make $A^\alpha \circledast B^\alpha = [a_1^\alpha, a_2^\alpha] \circledast [b_1^\alpha, b_2^\alpha] = [c_1^\alpha, c_2^\alpha]$, where the operation $*$ can be addition, subtraction, multiplication or division.
- From $\alpha = h_1(c_1^\alpha)$ and from $\alpha = h_2(c_2^\alpha)$ we obtain the membership functions of the resulted fuzzy number, on intervals: $y = h_1(x)$ and $y = h_2(x)$

# Examples for triangular fuzzy numbers

- A triangular fuzzy number is denoted $[m_1, m_2, m_3]$, where $m_1 \leq m_2 \leq m_3 \in \mathbb{R}$ (or $\mathbb{R}^+$ for multiplication and division) are the vertices of the triangle ($\mu(m_1) = \mu(m_3) = 0$ and $\mu(m_2) = 1$
- The sum and the difference of 2 triangular fuzzy numbers give a triangular fuzzy number,
- For addition, the coordinates of the sum are the sum of the coordinates with the same index
- Subtraction means the addition with the opposite number (the image), which is $[-m_3, -m_2, -m_1]$
- The coordinates for multiplication are computed similar to addition, i.e., by the multiplication of the coordinates with the same index, but the multiplication does not preserve the linearity.
- Division means the multiplication of the first number with the inverse of thhe second number, which is given by $[\frac{1}{m_3}, \frac{1}{m_2}, \frac{1}{m_1}]$

# Representation of triangular fuzzy numbers by membership functions

- If a triangular fuzzy number $A$ has the coordinates $[m_1, m_2, m_3]$, then its membership function is:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x \leq m_1 \\ \frac{x-m_1}{m_2-m_1}, & \text{if } m_1 \leq x \leq m_2 \\ \frac{m_3-x}{m_3-m_2}, & \text{if } m_2 \leq x \leq m_3 \\ 0, & \text{if } x \geq m_3 \end{cases}$$

- We want to obtain $A^\alpha$, where $A^\alpha = [a_1^\alpha, a_2^\alpha]$
- From $\frac{x-m_1}{m_2-m_1} = \alpha$ it results $x = m_1 + \alpha \cdot (m_2 - m_1)$
- Which means $a_1^\alpha = m_1 + \alpha \cdot (m_2 - m_1)$
- Similar, from $\frac{m_3-x}{m_3-m_2} = \alpha$ we obtain $x = m_3 - \alpha \cdot (m_3 - m_2)$
- which means $a_2^\alpha = m_3 - \alpha \cdot (m_3 - m_2)$

# Example of addition of two triangular fuzzy numbers

- Here we present the example 1.5, pp. 15–16, from [KG91]
- We add the triangular fuzzy numbers $A = [-5, -2, 1]$ and $B = [-3, 4, 12]$ and we will obtain the triangular fuzzy number $C = A(+)B = [-8, 2, 13]$
- The fuzzy number $A$ can be written:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x \leq -5 \\ \frac{x}{3} + \frac{5}{3}, & \text{if } -5 \leq x \leq -2 \\ -\frac{x}{3} + \frac{1}{3}, & \text{if } -2 \leq x \leq 1 \\ 0, & \text{if } x \geq 1 \end{cases}$$

- Making $\mu_A(x) = \alpha$ we obtain for $a_1^\alpha$: $\alpha = \frac{a_1^\alpha}{3} + \frac{5}{3}$ $\Rightarrow a_1^\alpha = 3 \cdot \alpha - 5$
- Similar, for $a_2^\alpha$: $\alpha = -\frac{a_2^\alpha}{3} + \frac{1}{3} \Rightarrow a_2^\alpha = -3 \cdot \alpha + 1$
- Hence $A^\alpha = [a_1^\alpha, a_2^\alpha] = [3\alpha - 5, -3\alpha + 1]$

# Example of addition of two triangular fuzzy numbers

- The fuzzy number $B$ can be written:

$$\mu_{\tilde{B}}(x) = \begin{cases} 0, & \text{if } x \leq -3 \\ \frac{x}{7} + \frac{3}{7}, & \text{if } -3 \leq x \leq 4 \\ -\frac{x}{8} + \frac{12}{8}, & \text{if } 4 \leq x \leq 12 \\ 0, & \text{if } x \geq 12 \end{cases}$$

- Making $\mu_B(x) = \alpha$ we obtain for $b_1^\alpha$: $\alpha = \frac{b_1^\alpha}{7} + \frac{3}{7}$ $\Rightarrow b_1^\alpha = 7 \cdot \alpha - 3$
- Similar, for $b_2^\alpha$: $\alpha = -\frac{b_2^\alpha}{8} + \frac{12}{8} \Rightarrow b_2^\alpha = -8 \cdot \alpha + 12$
- Hence $B^\alpha = [b_1^\alpha, b_2^\alpha] = [7\alpha - 3, -8\alpha + 12]$
- Then: $A^\alpha(+)B^\alpha = [3\alpha - 5, -3\alpha + 1](+)[7\alpha - 3, -8\alpha + 12]$
- Hence $C^\alpha = A^\alpha(+)B^\alpha = [10\alpha - 8, -11\alpha + 13]$

# Example of addition of two triangular fuzzy numbers

- From $C^{\alpha} = (c_1^{\alpha}, c_2^{\alpha}) = [10\alpha - 8, -11\alpha + 13]$ we obtain the membership function $\mu_C(x)$ of the fuzzy number $C$
- From $x = c_1^{\alpha} = 10\alpha - 8$ it results $\mu_C(x) = \alpha = \frac{x}{10} + \frac{8}{10}$ for $-8 \leq x \leq 2$
- From $x = c_2^{\alpha} = -11\alpha + 13$ it results $\mu_C(x) = \alpha = -\frac{x}{11} + \frac{13}{11}$ for $2 \leq x \leq 13$
- Hence, the fuzzy $C$ can be written:

$$\mu_{\tilde{C}}(x) = \begin{cases} 0, & \text{if } x \leq -8 \\ \frac{x}{10} + \frac{8}{10}, & \text{if } -8 \leq x \leq 2 \\ -\frac{x}{11} + \frac{13}{11}, & \text{if } 2 \leq x \leq 13 \\ 0, & \text{if } x \geq 13 \end{cases}$$

# Example of addition of two triangular fuzzy numbers

- **Comment:** The vertices of the triangular fuzzy number $C = A(+)B$ are obtained by making $\alpha$ equal with 0, respectively with 1 in the relations which describe $c_1^\alpha$ and $c_2^\alpha$:
- Making $\alpha = 0$, $10\alpha - 8$ becomes $-8$, and $-11\alpha + 13$ is equal with 13;
- For $\alpha = 1$ we obtain $10\alpha - 8 = -11\alpha + 13 = 2$
- Hence, as expected, the result of the addition is the triangular fuzzy number $[-8, -2, 13]$

# Example of multiplication of two triangular fuzzy numbers

- Based on the example 1.9, pp. 25–27, from [KG91], we multiply the triangular fuzzy numbers $A = [2, 3, 4]$ and $B = [3, 5, 6]$ and we will obtain the fuzzy number $C = A(\cdot)B = (6, 15, 24)$, which is not triangular (the membership function is not piecewise linear).
- We can write the fuzzy number $A$ as:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x \leq 2 \\ x - 2, & \text{if } 2 \leq x \leq 3 \\ 4 - x, & \text{if } 3 \leq x \leq 4 \\ 0, & \text{if } x \geq 4 \end{cases}$$

- Making $\mu_A(x) = \alpha$ we obtain for $a_1^\alpha$: $\alpha = a_1^\alpha - 2$ $\Rightarrow a_1^\alpha = \alpha + 2$
- For $a_2^\alpha$: $\alpha = 4 - a_2^\alpha \Rightarrow a_2^\alpha = 4 - \alpha$
- Hence $A^\alpha = [a_1^\alpha, a_2^\alpha] = [\alpha + 2, 4 - \alpha]$

# Example of multiplication of two triangular fuzzy numbers

- We can write the fuzzy number $B$ as:

$$\mu_{\tilde{B}}(x) = \begin{cases} 0, & \text{if } x \leq 3 \\ \frac{x}{2} - \frac{3}{2}, & \text{if } 3 \leq x \leq 5 \\ -x + 6, & \text{if } 5 \leq x \leq 6 \\ 0, & \text{if } x \geq 6 \end{cases}$$

- Making $\mu_B(x) = \alpha$ we obtain for $b_1^\alpha$: $\alpha = \frac{b_1^\alpha}{2} - \frac{3}{2}$, $\Rightarrow b_1^\alpha = 2 \cdot \alpha + 3$
- For $b_2^\alpha$: $\alpha = -b_2^\alpha + 6 \Rightarrow b_2^\alpha = -\alpha + 6$
- Hence $B^\alpha = [b_1^\alpha, b_2^\alpha] = [2\alpha + 3, 6 - \alpha]$
- Then: $A^\alpha(\cdot)B^\alpha = [\alpha + 2, 4 - \alpha](\cdot)[2\alpha + 3, 6 - \alpha]$
- Hence $C^\alpha = A^\alpha(\cdot)B^\alpha = [a_1^\alpha \cdot b_1^\alpha, a_2^\alpha \cdot b_2^\alpha]$
- $C^\alpha = [c_1^\alpha, c_2^\alpha] = [(\alpha + 2) \cdot (2\alpha + 3), (-\alpha + 4) \cdot (-\alpha + 6)]$
- $[c_1^\alpha, c_2^\alpha] = [2\alpha^2 + 7\alpha + 6, \alpha^2 - 10\alpha + 24]$

# Example of multiplication of two triangular fuzzy numbers

- In order to express $\alpha$ as a function of $c_1^\alpha$, we denote $c_1^\alpha$ with $x$ and we make $2\alpha^2 + 7\alpha + 6 = x$, i.e., we solve the second degree equqtion in $\alpha$: $2\alpha^2 + 7\alpha + 6 - x = 0$
- We recall that the second degree equation in $x$, $ax^2 + bx + c = 0$ has the solutions $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$
- In our case $\Delta = 7^2 - 4 \cdot 2 \cdot (6 - x) = 49 - 48 + 8x = 1 + 8x$
- Hence $\alpha_{1,2} = \frac{-7 \pm \sqrt{1+8x}}{4}$
- Since we discuss about fuzzy membership functions (with values in the interval $[0, 1]$, we consider only the solution situated in the real numbers interval $[0, 1]$, which is $\alpha = \frac{-7 + \sqrt{1+8x}}{4}$, for $6 \leq x \leq 15$

# Example of multiplication of two triangular fuzzy numbers

- In order to express $\alpha$ as a function of $c_2^\alpha$, we denote $c_2^\alpha$ with $x$ and we make $\alpha^2 - 10\alpha + 24 = x$, i.e., we solve the second degree equation in $\alpha$: $\alpha^2 - 10\alpha + 24 - x = 0$
- The solution is $\alpha_{1,2} = 5 \pm \sqrt{1+x}$
- We chose the solution situated in the interval $[0,1]$, which is $\alpha = 5 - \sqrt{1+x}$, where $15 \leq x \leq 24$
- In conclusion, the result of the multiplication is the fuzzy number (fuzzy set) $C$:

$$\mu_{\tilde{C}}(x) = \begin{cases} 0, & \text{if } x \leq 6 \\ -\frac{7}{4} + \frac{\sqrt{1+8x}}{4}, & \text{if } 6 \leq x \leq 15 \\ 5 - \sqrt{1+x}, & \text{if } 15 \leq x \leq 24 \\ 0, & \text{if } x \geq 24 \end{cases}$$

# Example of division of two triangular fuzzy numbers

- We show example 1.11, pp. 32–33, from [KG91], where the triangular fuzzy number $A = [18, 22, 33]$ is divided to the triangular fuzzy number $B = [5, 6, 8]$, and we obtain the fuzzy number $C = A(:)B = [18/8, 22/6, 33/5]$, which is not a triangular fuzzy number (its membership function is not piecewise linear)
- The fuzzy number $A$ can be written:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if } x \leq 18 \\ \frac{x}{4} - \frac{18}{4}, & \text{if } 18 \leq x \leq 22 \\ -\frac{x}{11} + \frac{33}{11}, & \text{if } 22 \leq x \leq 33 \\ 0, & \text{if } x \geq 33 \end{cases}$$

- Making $\mu_A(x) = \alpha$, we obtain for $a_1^\alpha$: $\alpha = \frac{a_1^\alpha}{4} - \frac{18}{4}$, $\Rightarrow a_1^\alpha = 4\alpha + 18$
- Similar, for $a_2^\alpha$: $\alpha = -\frac{a_2^\alpha}{11} + \frac{33}{11} \Rightarrow a_2^\alpha = -11\alpha + 33$
- Hence $A^\alpha = [a_1^\alpha, a_2^\alpha] = [4\alpha + 18, -11\alpha + 33]$

# Example of division of two triangular fuzzy numbers

- The fuzzy number $B$ can be written:

$$\mu_{\tilde{B}}(x) = \begin{cases} 0, & \text{if } x \leq 5 \\ x - 5, & \text{if } 5 \leq x \leq 6 \\ -\frac{x}{2} + \frac{8}{2}, & \text{if } 6 \leq x \leq 8 \\ 0, & \text{if } x \geq 8 \end{cases}$$

- Making $\mu_B(x) = \alpha$, we obtain for $b_1^\alpha$: $\alpha = b_1^\alpha - 5$, $\Rightarrow b_1^\alpha = \alpha + 5$
- Similar, for $b_2^\alpha$: $\alpha = -\frac{b_2^\alpha}{2} + \frac{8}{2}$, $\Rightarrow b_2^\alpha = -2\alpha + 8$
- Hence $B^\alpha = [b_1^\alpha, b_2^\alpha] = [\alpha + 5, -2\alpha + 8]$
- Then: $A^\alpha(:)B^\alpha = [4\alpha + 18, -11\alpha + 33](:)[\alpha + 5, -2\alpha + 8]$
- Hence $C^\alpha = A^\alpha(:)B^\alpha = [a_1^\alpha : b_2^\alpha, a_2^\alpha : b_1^\alpha]$
- $C^\alpha = [c_1^\alpha, c_2^\alpha] = [\frac{4\alpha+18}{-2\alpha+8}, \frac{-11\alpha+33}{\alpha+5}]$

# Example of division of two triangular fuzzy numbers

- In order to find the membership function of the fuzzy number $C$, we denote $c_1^\alpha = x$ and make $\frac{4\alpha+18}{-2\alpha+8} = x$, then we express $\alpha$ as a function of $x$:
- $\frac{4\alpha+18}{-2\alpha+8} = x \Rightarrow 4\alpha + 18 = -2x\alpha + 8x \Rightarrow 4\alpha + 2x\alpha = 8x - 18$
- $\alpha(2x+4) = 8x - 18$
- $\alpha = \frac{8x-18}{2x+4}$
- Similar, we denote $c_2^\alpha = x$ and it results $\frac{-11\alpha+33}{\alpha+5} = x$ $\Rightarrow -11\alpha + 33 = \alpha \cdot x + 5x$
- $-\alpha x - 11\alpha = 5x - 33$
- $\alpha(x+11) = -5x + 33$
- $\alpha = \frac{-5x+33}{x+11}$

# Example of division of two triangular fuzzy numbers

- Hence, the fuzzy number $C = A(:)B$ can be written:

$$\mu_{\tilde{C}}(x) = \begin{cases} 0, & \text{if } x \leq 9/4 \\ \frac{8x-18}{2x+4}, & \text{if } 9/4 \leq x \leq 11/3 \\ \frac{-5x+33}{x+11}, & \text{if } 11/3 \leq x \leq 33/5 \\ 0, & \text{if } x \geq 33/5 \end{cases}$$

- In order to obtain the left and right vertices of the triangular fuzzy number $C$ we make $\alpha = 0$ and we obtain:
  $8x - 18 = 0 \Rightarrow x = 18/8 = 9/4$, respective
  $-5x + 33 = 0 \Rightarrow x = 33/5$
- In order to obtain the middle vertice of the triangular fuzzy number $C$ we make $\alpha = 1$, i.e.
  $\frac{8x-18}{2x+4} = 1 \Rightarrow 8x - 18 = 2x + 4, \Rightarrow 6x = 22$
  $\Rightarrow x = 22/6 = 11/3$
- Verification: $\frac{-5x+33}{x+11} = 1 \Rightarrow -5x + 33 = x + 11 \Rightarrow -6x = -22 \Rightarrow x = 22/6 = 11/3$, correct !

Arnold Kaufman and Madan M Gupta.
*Introduction to fuzzy arithmetic*.
Van Nostrand Reinhold Company New York, 1991.