Fuzzy logic. Approximate reasoning. **Fuzzy inference**

Chapter 5

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Outline

Linguistic variables

Fuzzy logic

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Linguistic variables

They have been introduced by Zadeh in 1973.

Zadeh said “in retreating from precision in the face of overpowering complexity, it is natural to explore of what might be called *linguistic variables*, that is, variables whose values are not numbers but words or sentences in a natural or artificial language. The motivation for the use of words or sentences rather than numbers is that linguistic characterizations are, in general, less specific than numerical ones.” [Zim91], p. 131.

Linguistic variables. Definitions

Definition

A linguistic variable is characterized by a quintuple $(x,\ T(x),\ U,\ G,\ \tilde{M})$, where:

- x is the name of the linguistic variable
- $T(x)$ is the *(term set)* of the linguistic variable
- U is the universe of discourse associated to the base variable u
- G is a set of syntactic rules for generating the terms (usually it is a grammar)
- $\tilde{M}$ is a set of semantic rules that associate to each term a meaning;
  $\tilde{M}$ is a fuzzy set in U

Linguistic variables. Example

Linguistic variable *age:*

- $x = age$
- $T(x) =$ $\{very\ young, young, more\ or\ less\ young, old, very\ old\}$
- $U = [0, 100]$ age expressed in years, in the real numbers interval $[0, 100]$
- We give an example for $\tilde{M}(old)$, which is a fuzzy set in $u \in U$ given by:

$$\tilde{M}(old) = \{(u, \mu_{old}(u)) \mid u \in [0, 100]\}$$

where $\mu_{old}(u)$ can be given by the formula:

$$\mu_{old}(u) = \begin{cases} 0, & \text{if } x < 50 \\ [1 + (\frac{u-50}{5})^{-2}]^{-1}, & \text{if } x \in [50, 100] \end{cases}$$

We obtain $\mu_{old}(u) = \begin{cases} 0, & \text{if } x < 50 \\ \frac{(u-50)^2}{(u-50)^2+5^2}, & \text{if } x \in [50, 100] \end{cases}$

Figure !

Linguistic modifiers

Definition

A *linguistic hedge (linguistic modifier)* is an operation that modifies the meaning of a fuzzy term, or, more general, the meaning of a fuzzy set.
If $\tilde{A}$ is a fuzzy set, then the modifier m generates the fuzzy set $\tilde{B} = m(\tilde{A})$

Example of linguistic modifiers. Given the fuzzy set $\tilde{A}$, with $\mu_{\tilde{A}}(u)$, it can be associated the following linguistic modifiers:

1. *(concentration)*: $\mu_{con(\tilde{A})}(u) = (\mu_{\tilde{A}}(u))^2$
2. *(dilation)*: $\mu_{dil(\tilde{A})}(u) = (\mu_{\tilde{A}}(u))^{1/2}$
3. *(contrast intensification)*:
$$\mu_{int(\tilde{A})}(u) = \begin{cases} 2 \cdot (\mu_{\tilde{A}}(u))^2, \text{ if } \mu_{\tilde{A}}(u) \in [0, 0.5] \\ 1 - 2 \cdot (1 - \mu_{\tilde{A}}(u))^2, \text{ if } \mu_{\tilde{A}}(u) \in [0.5, 1] \end{cases}$$

The meaning of linguistic modifiers

Given a fuzzy set $\tilde{A}$ with the membership function $\mu_{\tilde{A}}(u)$ the linguistic modifiers are usually associated the following meaning:

- $\textit{very}\ \tilde{A} = con(\tilde{A})$
- $\textit{more or less}\ \tilde{A} = dil(\tilde{A})$
- $\textit{plus}\ \tilde{A} = (\mu_{\tilde{A}}(u))^{1.25}$
- $\textit{minus}\ \tilde{A} = (\mu_{\tilde{A}}(u))^{0.25}$

Structured and Boolean linguistic variables

Definition

A linguistic variable x is *structured* if $T(x)$ and $\tilde{M}(x)$ can be obtained algorithmically. The algorithms imply the utilization of linguistic modifiers, for generating the terms of the linguistic variable.

Example:

- $T^0 = \emptyset$
- $T^1 = \{old\}$
- $T^2 = \{old, \textit{very old}\}$
- $T^2 = \{old, \textit{very very old}\}$

Definition

A linguistic variable is called *Boolean* if its term set can be obtained from a primary term set to which are applied linguistic modifiers and Boolean operations (NOT, AND, OR).

Special linguistic variables

1. *Probability* =
   {*almost impossible*, *not very probable*, *very probable*, *almost certain*}
   Figure !
2. Truth. Pt *truth* there are two representations (definitions), one from Zadeh and one from Baldwin:
   2.1 Zadeh
   2.2 Baldwin

Linguistic variable *truth* in Zadeh's definition:
For $u \in [0, 1]$ we define:

$$\mu_{true}(u) = \begin{cases} 0, \text{ if } u \leq a \\ 2 \cdot (\frac{u-a}{1-a})^2, \text{ if } a \leq u \leq \frac{a+1}{2} \\ 1 - 2 \cdot (\frac{u-a}{1-a})^2, \text{ if } \frac{a+1}{2} \leq u \leq 1 \end{cases}$$

$\mu_{false}(u) = \mu_{true}(1 - u)$
$a \in [0, 1]$ is called *crossover point* and it indicates subjectivity.

Special linguistic variables

Linguistic variable *truth* in Baldwin's definition:
For $u \in [0,1]$ we define:

- $\mu_{true}(u) = u$
- $\mu_{false}(u) = 1 - u = \mu_{true}(1 - u)$
- $\mu_{very\ \ true}(u) = (\mu_{true}(u))^2 = u^2$
- $\mu_{very\ \ false}(u) = (\mu_{false}(u))^2 = (1 - u)^2$
- $\mu_{fairly\ \ true}(u) = (\mu_{true}(u))^{1/2} = u^{1/2}$
- $\mu_{fairly\ \ false}(u) = (\mu_{false}(u))^{1/2} = (1 - u)^{1/2}$

Figure for truth in Zadeh's and Baldwin's definitions !!!

Outline

Classic logic revisited

Classic logic relies on 3 items:

1. Truth values:
   - 0 or *false*
   - 1 or *true*
2. Vocabulary (operators): AND, OR, NOT, XOR, NAND, NOR, $\Rightarrow$ (implication), $\Leftrightarrow$ (equivalence)
3. Reasoning procedures (tautologies): statements that are always true.

We want to extend these elements to multivalued logic.

Logic operators in classic logic

- In classic logic, the operators are defined by tables.
- Assuming that we have two statements, p and q, each of them can have either the value 0, or the value 1.
- It means that we can have $2 \times 2 = 4$ combinations for the values of p and q, i.e., a table with 4 lines.
- With these 4 combinations for p and q we can have $2^4 = 16$ possible combinations, i.e. 16 columns in the table.
- Each of the 16 columns is an operator
- Some operators have a name associated with them (p AND q), (NOT p OR q, i.e. implication $p \Rightarrow q$), etc, but it is difficult to associate a name to all columns

Logic operators in classic logic

p	q	$\wedge$	$\vee$	XOR	$\Rightarrow$	$\Leftrightarrow$	?
0	0	0	0	0	1	1	0
0	1	0	1	1	1	0	0
1	0	0	1	1	0	0	1
1	1	1	1	0	1	1	1

Table 1: Logic operators in classic logic

Symbol $\wedge$ means AND, symbol $\vee$ represents OR, symbol $\neg$ represents NOT.
$\neg p \vee q = p \Rightarrow q$ (not p or q, which is p implies q)
$(\neg p \vee q) \wedge (\neg q \vee p) = p \Leftrightarrow q$ (p is equivalent with q, i.e. p implies q and q implies p)

Extension to multivalued logic

- If instead of two logic values, we would have 3 logic (truth) values: $\{0, \frac{1}{2}, 1\}$
- Then, each of the sentences p and q could have 3 truth values
- Hence, the operators table would have $3 \times 3 = 9$ lines
- Number of possible columns will be 3^9 !!!
- We can however define a three-valued logic, i.e. logic with 3 truth values: $\{F, T + F, T\}$ (false, true plus false, i.e. undecided, and true)
- With these truth values the operators NOT, AND and OR can be defined as follows:

	$\neg$
T	F
F	T
T+F	T+F

Table 2: Function NOT in three-valued logic

Extension to multivalued logic

$\wedge$	T	F	T+F
T	T	F	T+F
F	F	F	F
T+F	T+F	F	T+F

Table 3: Function AND in three-valued logic

$\vee$	T	F	T+F
T	T	T	T
F	T	F	T+F
T+F	T	T+F	T+F

Table 4: Function OR in three-valued logic

Tautologies in classic logic

Following tautologies are used in classic logic:

1. modus ponens: $A \wedge (A \Rightarrow B) \Rightarrow B$
   If A is true and if the statement "A implies B" is true, then B is true
   The part "A is true" is the first hypothesis (hypothesis 1), part "A implies B" is hypothesis 2, and "B is true" is the conclusion
2. modus tollens: $((A \Rightarrow B) \wedge (\neg B)) \Rightarrow \neg A$
   If A implies B and if B is false, then A is false
3. syllogism: $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
   If A implies B and if B implies C, then A implies C
4. Contraposition: $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
   If A implies B, then NOT A implies NOT B

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Approximate reasoning

Approximate reasoning

The following slides contain text, figures, formulas, etc, taken (and modified) from the final year project of Ana-Maria Badulescu [Bad99], which were based on, or taken from Zimmermann [Zim91] and Chiueh [Chi92].

Approximate reasoning means fuzzy inference.

In fuzzy set theory, modus ponens has been generalized by Zadeh, Mizumoto, Mamdani as follows: if $\tilde{A}$, $\tilde{A}'$, are fuzzy sets in X, and $\tilde{B}$, $\tilde{B}'$ are fuzzy sets in Y, then generalized modus ponens is given in the following table:

Premise	x is $\tilde{A}'$
Implication	If x is $\tilde{A}$ then y is $\tilde{B}$
Conclusion	y is $\tilde{B}'$

Table 5: Generalized modus ponens

The part of the rule between IF and THEN is *antecedent* or *premises*, and the part of the rule after THEN is called *conclusion* or *consequent*, while $\tilde{A}'$ is the input *fact*.

Generalized modus ponens : example

An example of generalized modus ponens is given in the following table:

Premise	This banana is very yellow
Implication	If a banana is yellow, then the banana is ripe
Conclusion	This banana is very ripe

Fuzzy inference

In order to find a mathematical representation of these logic expressions, researchers proposed different mathematical formulas (maximum, minimum, scalar product) for the logical and implication operators.
Different researchers gave different mathematical expressions, the most used being the compositional rule of inference, proposed by Zadeh and Mamdani.
The formula for generalized modus ponens is:

$$\tilde{B}' = \tilde{A}' \bullet \tilde{R}_{\tilde{A} \to \tilde{B}} \tag{1}$$

The implication $\tilde{A} \to \tilde{B}$ is a fuzzy relation, having the membership function $\mu_{\tilde{R}_{\tilde{A} \to \tilde{B}}}(x, y)) = \min(\mu_{\tilde{A}(x), \mu_{\tilde{B}}}(y))$
For the composition operator of the premise $\tilde{A}'$ with the implication, Zadeh proposed the expression (used at the max-min composition of fuzzy relations):

$$\max_x \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{R}_{\tilde{A} \to \tilde{B}}}(x, y)\} \tag{2}$$

Fuzzy inference

The membership function of $\tilde{B}'$:

$$\mu_{\tilde{B}'}(y) = \max_{x \in X} \min(\mu_{\tilde{A}'}(x), \mu_{\tilde{R}_{\tilde{A} \to \tilde{B}}}(x, y)) =$$

$$= \max_{x \in X} \min(\mu_{\tilde{A}'}(x), \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y))) =$$

$$= \min \max_{x \in X} [\min(\mu_{\tilde{A}'}(x), \mu_{\tilde{A}}(x)), \mu_{\tilde{B}}(y)] = \min(\Omega, \mu_{\tilde{B}}(y))$$

where

$$\Omega = \max_{x \in X} \min(\mu_{\tilde{A}'}(x), \mu_{\tilde{A}}(x))$$

is called degree of activation (or firing strength) of the rule (sometimes it is denoted with α or with a).

Fuzzy inference: graphical example

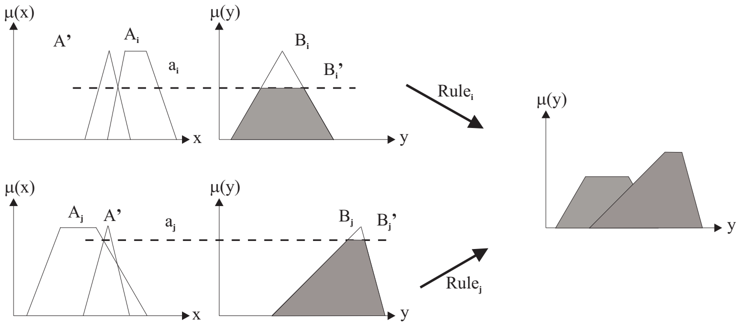

Figure 1: Example of fuzzy inference

Ana-Maria Badulescu.
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