Fuzzy relations

Chapter 4

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Outline

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Definitions

- In this course we will discuss only binary fuzzy relations
- The generalization to *n*-ary fuzzy relations is straightforward
- Fuzzy relations are subsets of the cartesian product $X \times Y$, where X and Y are universes of discourse.

Definition

Given the universes of discourse X and Y, a fuzzy relation $\tilde{R}$ in $X \times Y$ is defined as the set
$\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$, where
$\mu_{\tilde{R}}(x, y) : X \times Y \to [0, 1]$

Examples of fuzzy relations

1. For $X = Y = \mathbb{R}$, we define the continuous fuzzy relation “x considerably larger than y” :
$$\mu_{\tilde{R}}(x,y) = \begin{cases} 0, \text{ if } x \leq y \\ \frac{|x-y|}{10\cdot|y|}, \text{ if } y < x \leq 11\cdot y \\ 1, \text{ if } x > 11\cdot y \end{cases}$$
2. The fuzzy relation “$x \gg y$” could be defined also as:
$$\mu_{\tilde{R}}(x,y) = \begin{cases} 0, \text{ if } x \leq y \\ \frac{(x-y)^2}{1+(x-y)^2}, \text{ if } x > y \end{cases}$$

Examples of fuzzy relations

3. For the discrete fuzzy sets $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, the fuzzy relation $\tilde{R}$ "$x \gg y$" can be expressed by the matrix:

	y_1	y_2	y_3
x_1	0.5	1	0
x_2	0.7	0.2	0.1

Table 1: Fuzzy relation $\tilde{R}$

Definition

Definition

Let $X, Y \subseteq \mathbb{R}$ and

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$$

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y \in Y\}$$

Then $\tilde{R} = \{(x, y, \mu_{\tilde{R}}(x, y)) | (x, y) \in X \times Y\}$ is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{A}}(x), \ \forall (x, y) \in X \times Y$$

si

$$\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{B}}(y), \ \forall (x, y) \in X \times Y$$

.

This definition is useful for fuzzy graphs, but we will not study fuzzy graphs in this course.

Union and intersection of fuzzy relations

Being fuzzy sets, between fuzzy relations defined on the same universes of discourse we can perform union and intersection operations as follows:

Definition

Let $\tilde{R}$ and $\tilde{Z}$ two fuzzy relations defined in the same product space. Union, respectively intersection of the fuzzy relations $\tilde{R}$ and $\tilde{Z}$ are defined:

$$\mu_{\tilde{R}\cup\tilde{Z}}(x,y) = \max\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y)\}, \quad (x,y) \in X \times Y$$

$$\mu_{\tilde{R}\cap\tilde{Z}}(x,y) = \min\{\mu_{\tilde{R}}(x,y), \mu_{\tilde{Z}}(x,y)\}, \quad (x,y) \in X \times Y$$

The projections of a fuzzy relation

Definition

Let $\tilde{R} = \{(x, y, \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$ be a binary fuzzy relation. *The first projection* of the relation $\tilde{R}$ is defined:

$$\tilde{R}^{(1)} = \{(x, \max_y \mu_{\tilde{R}}(x, y) \mid (x, y) \in X \times Y\}$$

The second projection is defined:

$$\tilde{R}^{(2)} = \{(y, \max_x \mu_{\tilde{R}}(x, y) \mid (x, y) \in X \times Y\}$$

The total projection is defined:

$$\tilde{R}^{(T)} = \max_x \max_y \{\mu_{\tilde{R}}(x, y) \mid (x, y) \in X \times Y\}$$

The first projection is a function in x, the second projection is a function in y, while the total projection is a number (a single value).

The cylindrical extension of a fuzzy relation

Definition

Given a fuzzy relation $\tilde{R}$ who's projection (first or second) is $\tilde{R}_q$, the *cylindrical extension* $\tilde{R}_{qL}$ of $\tilde{R}$ is the smallest relation who's projection is $\tilde{R}_q$, and $\tilde{R}_q$ is *the base* of $\tilde{R}_{qL}$.

Remark: Zimmermann [Zim91] defines the cylindrical extension for n-ary relation, not only for binary fuzzy relations.

Next slide presents an example with first, second, and total projections and the cylindrical extension of a binary discrete fuzzy relation.

Remark: The α cuts can be defined for fuzzy relations as well (not only for fuzzy sets), e.g. in [NR74].

Example of projections for a discrete fuzzy relation

Consider $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. In table 2 is given the fuzzy relation $R = X \times Y$, its first projection $R^{(1)}$, its second projection $R^{(2)}$, and the total projection $R^{(T)}$:

	y_1	y_2	y_3	y_4	$R^{(1)} = \max_y$
x_1	0.2	0.7	1	0.5	1
x_2	0.3	0.8	0	0.2	0.8
x_3	0.5	0.1	0.7	0.9	0.9
$R^{(2)}$	0.5	0.8	1	0.9	$R^{(T)} = 1$

Table 2: The fuzzy relation $\tilde{R}$ and its projections

	y_1	y_2	y_3	y_4
x_1	0.5	0.8	1	0.9
x_2	0.5	0.8	1	0.9
x_3	0.5	0.8	1	0.9

Table 3: The cylindrical extension of the fuzzy relation $\tilde{R}^{(2)}$

Outline

Composition of fuzzy relations

- Discrete fuzzy relations, described by matrixes, can be composed in a similar way with matrixes multiplication
- Like in the case of matrixes, in order to compose the fuzzy relations, their dimensions have to match
- The most used composition method is the $\max - \min$ composition
- Given two fuzzy relations expressed in matrix form, after the *max-min* composition it results a new fuzzy relation (a matrix) whose elements are obtained by "multiplication" of the elements of the two relations
- "Multiplication" is made between a line from the first matrix and a column from the second matrix, but instead of $+$ and $\cdot$ we use max and min
- There exists other composition methods than *max-min*, more precisely the *min* operation can be replaced by algebraic product, average, or other operations.
- Composition of fuzzy relations is important for understanding fuzzy inference.

The max-min composition

Definition

Given the fuzzy relations $\tilde{R}_1(x, y) \subset X \times Y$ and $\tilde{R}_2(y, z) \subset Y \times Z$, their max-min composition, $\tilde{R}_1$ *max-min* $\tilde{R}_2$, denoted $\tilde{R}_1 \circ \tilde{R}_2$ is defined as the fuzzy set:

$$\tilde{R}_1 \circ \tilde{R}_2 = \{((x, z), \mu_{\tilde{R}_1 \circ \tilde{R}_2}(x, z)) \mid (x, z) \in X \times Z\}$$

, where

$$\mu_{\tilde{R}_1 \circ \tilde{R}_2}(x, z) = \max_y [\min(\mu_{\tilde{R}_1}(x, y), \mu_{\tilde{R}_2}(y, z))]$$

The max-star composition

Definition

Given the fuzzy relations $\tilde{R}_1(x, y) \subset X \times Y$ and $\tilde{R}_2(y, z) \subset Y \times Z$, their max-star composition, $\tilde{R}_1 \circledast \tilde{R}_2$ is defined as the fuzzy set:

$$\tilde{R}_1 \circledast \tilde{R}_2 = \{((x, z), \mu_{\tilde{R}_1 \circledast \tilde{R}_2}(x, z)) \mid (x, z) \in X \times Z\}$$

, where

$$\mu_{\tilde{R}_1 \circledast \tilde{R}_2}(x, z) = \max_y [(\mu_{\tilde{R}_1}(x, y) * \mu_{\tilde{R}_2}(y, z))]$$

If the operation $*$ (*star*) is associative and monotonically nondecreasing in each argument, then the $\max - *$ composition has similar properties with the $\max - \min$ composition.
The most employed are the *max-prod* $(\max - \cdot)$(when the operation $*$ is the algebraic product, and *max-average*, when operation $*$ is the arithmetic mean.

Examples of compositions of discrete binary fuzzy relations

Let $\tilde{R}_1(x, y)$ and $\tilde{R}_2(y, z)$ discrete binary fuzzy relations defined by the following matrices:

$\tilde{R}_1$:

	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.2	0	1	0.7
x_2	0.3	0.5	0	0.2	1
x_3	0.8	0	1	0.4	0.3

$\tilde{R}_2$:

	z_1	z_2	z_3	z_4
y_1	0.9	0	0.3	0.4
y_2	0.2	0.1	0.8	0
y_3	0.8	0	0.7	1
y_4	0.4	0.2	0.3	0
x_5	0	1	0	0.8

Examples of compositions of discrete binary fuzzy relations

The results of max-min, max-prod and max-average composition of the relations $\tilde{R}_1(x, y)$ and $\tilde{R}_2(y, z)$ are given by the following matrices:

$\tilde{R}_1 \ \max - \min \ \tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.7
x_2	0.3	1	0.5	0.8
x_3	0.8	0.3	0.7	1

$\tilde{R}_1 \ \max - \cdot \ \tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.56
x_2	0.27	1	0.4	0.8
x_3	0.8	0.3	0.7	1

Examples of compositions of discrete binary fuzzy relations

$\tilde{R}_1$ *max-average* $\tilde{R}_2$

	z_1	z_2	z_3	z_4
x_1	0.7	0.85	0.65	0.75
x_2	0.6	1	0.65	0.9
x_3	0.9	0.65	0.85	1

Outline

Properties of max-min composition

1. *Associativity* The *max-min* composition is associative, that is

$$(\tilde{R}_3 \circ \tilde{R}_2) \circ \tilde{R}_1 = \tilde{R}_3 \circ (\tilde{R}_2 \circ \tilde{R}_1)$$

   Consequence: the power of a fuzzy relation. If
   $\tilde{R}_1 = \tilde{R}_2 = \tilde{R}_3 = \tilde{R}$, we can write:
   $\tilde{R}^2 = \tilde{R} \circ \tilde{R}$,
   $\tilde{R}^3 = \tilde{R} \circ \tilde{R} \circ \tilde{R}$ and so on.
   Hence, for any n natural, we can define $\tilde{R}^n$

2. *Reflexivity Definition:* A fuzzy relation defined in $X \times X$ is reflexive iff (if and only if) $\mu_{\tilde{R}}(x, x) = 1, \forall x \in X$
   Property: If the fuzzy relations $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive, then their max-min composition, $\tilde{R}_1 \circ \tilde{R}_2$ is also reflexive.

Properties of max-min composition

3. *Symmetry Definition:* A fuzzy relation is called symmetric if $\tilde{R}(x, y) = \tilde{R}(y, x)$
4. *Antisymmetry Definition:* A fuzzy relation is called antisymmetric if, $\forall x, y \in X$, if $x \neq y$ then either $\mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x)$, or $\mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(y, x) = 0$
5. A fuzzy is called perfect antisymmetric if $\forall\ x \neq y$, whenever $\mu_{\tilde{R}}(x, y) > 0$, then $\mu_{\tilde{R}}(y, x) = 0$
6. *Transitivity Definition:* A fuzzy relation $\tilde{R}$ is called max-min transitive if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$.

Properties of max-min composition

For the max-min composition, the following properties take place:

1. If $\tilde{R}_1$ is reflexive and $\tilde{R}_2$ is an arbitrary fuzzy relation, then $\tilde{R}_1 \circ \tilde{R}_2 \supseteq \tilde{R}_2$ si $\tilde{R}_2 \circ \tilde{R}_1 \supseteq \tilde{R}_2$
2. If $\tilde{R}$ is reflexive, then $\tilde{R} \subseteq \tilde{R} \circ \tilde{R}$
3. If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive, then their max-min composition $\tilde{R}_1 \circ \tilde{R}_2$ is also reflexive
4. If $\tilde{R}_1$ and $\tilde{R}_2$ are symmetric, then $\tilde{R}_1 \circ \tilde{R}_2$ is symmetric if $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$
5. If $\tilde{R}$ is symmetric, so is each power of $\tilde{R}$

Properties of max-min composition

Considering also transitivity, the following properties can be proved:

1. If a fuzzy relation is symmetric and transitive, then $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{R}}(x, x)$, $\forall\ x, y \in X$
2. If $\tilde{R}$ is reflexive and transitive, then $\tilde{R} \circ \tilde{R} = \tilde{R}$
3. If $\tilde{R}_1$ and $\tilde{R}_2$ are transitive and if $\tilde{R}_1 \circ \tilde{R}_2 = \tilde{R}_2 \circ \tilde{R}_1$, then $\tilde{R}_1 \circ \tilde{R}_2$ is transitive

Similarity, de pre-order and order fuzzy relations

Definition

A *similarity relation* is a fuzzy relation that is reflexive, symmetrical and max-min transitive

The idea of similarity is analogous to the idea of equivalence, being possible to create similarity trees.

Definition

A fuzzy relation which is max-min transitive and reflexive is called *fuzzy preorder relation*.

Definition

A fuzzy relation which is max-min transitive, reflexive and antisymmetric is called *fuzzy order relation*.

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